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# NONLINEAR STRESS-STRAIN AND INELASTIC REBOUND WITH EXPONENTIAL STIFFNESS

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U.S. ARMY LABORATORY COMMAND

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Bilinear or power-law representations are often used to describe the application and release of tension or compression. Here a new nonlinear equation of motion with exact solutions is developed which can describe not only a tension or compression half-cycle, but the entire hysteretic loop. Solutions of the same equation can be used to describe impact phenomena which are important in a number of areas from package cushioning to stopping a fragment or bullet. Oscillatory solutions can also be applied to a variety of mechanical and electrical phenomena. Specific examples are given of some two-segment limit cycles which are special cases of more general solutions. Methods of generalizing the approach used to many other cases are also indicated.

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# I. INTRODUCTION

When we model shock and impact attenuation we should include nonlinear forcing and damping terms[1]. After considering some of the models which have been used in the past, the author studied some new nonlinear equations which have applications in many fields besides mechanics. Because of their general interest, some of the solutions of one equation will be described in this report.

The study of nonlinear differential equations is difficult and few exact solutions are known for problems of general interest. When exact solutions are available, they offer several advantages. First of all, they simplify tasks like mapping stable regions, establishing the existence of limit cycles, and so on. In addition, it becomes easier to survey various combinations of parameters over the entire ranges of their allowed values. This can be a fruitful source of ideas. Finally, by avoiding approximations, we eliminate an undesirable source of uncertainty. With this in mind we will restrict ourselves in this report to exact solutions which can be expressed by a finite number of elementary functions.

The method we will follow consists of starting with an equation which has known solutions, transforming it in a general way and then choosing a particular form of interest. Since there can be many starter equations, transformations and particular forms, the case we will discuss is merely an example. The method is not new, but as we shall see, it can lead to new results of general interest.

Let us begin with the damped, driven linear oscillator

$$\ddot{X} + \beta\dot{X} + \omega^2 X = F \tag{1}$$

where $\beta$ and $\omega^2$ are constants. Solutions and applications of this equation are well known. Next, we choose a general transformation of the dependent variable by considering $X=X[x(t)]$ such that

$$\dot{X} = \dot{x}X' \quad \text{and} \quad \ddot{X} = \ddot{x}X' + \dot{x}^2 X'' \tag{2}$$

where a prime denotes differentiation with respect to the new variable x. Eq (1) becomes

$$X'\ddot{x} + X''\dot{x}^2 + \beta X'\dot{x} + \omega^2 X = F \quad . \tag{3}$$

Finally, we choose a particular form

$$X = \xi \exp(\alpha x) + \zeta \tag{4}$$

where $\alpha$, $\xi$ and $\zeta$ are constants. Putting Eq (4) in Eq (3) gives

$$\ddot{x} + \alpha\dot{x}^2 + \beta\dot{x} + (\omega^2/\alpha)[1 + (\zeta/\xi)\exp(-\alpha x)] = [F/(\alpha\xi)]\exp(-\alpha x) \tag{5}$$

which describes a forced nonlinear oscillator with quadratic as well as linear damping. Because of the forms in Eqs (4) and (5) we call this an exponential oscillator. Equations with linear rather than exponential stiffness have been discussed previously in connection with impact attenuation using either quadratic damping[2] or combined Coulomb, linear and quadratic damping[3]. Equation (5) is not as general as one might like since the quadratic damping coefficient, $\alpha$, appears in several places. However, this limitation can be advantageous if we desire a simple reduction to the linear case. For example, if $\xi=-\varsigma=(1/\alpha)$, then $x=(1/\alpha)\ln(1+\alpha X) \to X$ and Eq (5) $\to$ Eq (1) as $\alpha \to 0$.

Damping terms which depend on odd powers of $\dot{x}$ like $\beta\dot{x}$ automatically follow the sign of $\dot{x}$. If $\beta>0$, this term always opposes the motion (positive damping). If $\beta<0$, it always aids the motion (negative damping). However, terms which depend on even powers of $\dot{x}$ like $\alpha\dot{x}^2$ must have the sense of the damping specified. Various conventions have been used for particular applications. Here we will find it more convenient to specify the sign of $\alpha$ for each segment of a motion.

When the linear oscillator is externally excited by F(t) in Eq (1) we may combine the right side of Eq (5) with the last term on the left side to make it clear that the exponential oscillator is parametrically or internally excited. Various other forms of F may also be used. For example, if $F=\varphi\alpha[\xi \exp(\alpha x)]=\varphi\alpha(X-\varsigma)$ from Eq (4), the $\varphi\alpha X$ term may be combined with the $\omega^2 X$ term in Eq (1), leaving $-\varphi\alpha\varsigma$ on the right as a driving term. The right side of Eq (5) becomes $\varphi$. If $\varphi$ is a constant, the solutions are simple. If $\varphi$ is a sinusoidal function of time, Eq (1) becomes a damped, driven Hill's equation, while Eq (5) is externally excited, etc.

For constant $F=F_0$ the right side of equation (5) combined with the last part of the left side gives the stiffness or restoring force

$$f = (\omega^2/\alpha)(1+\delta \exp[-\alpha x]) \tag{6}$$

where $\delta=(\varsigma-F_0/\omega^2)/\xi$. For $\delta=-1$, a plot of f versus x will always pass through the origin. This includes the linear form $\omega^2 x$ which results when $\alpha\to 0$. For other values of $\alpha$, the curves exhibit "hardening" or "softening" and approach asymptotes which depend on the sign and magnitude of $\alpha$ for given $\omega^2$. For $\delta\neq-1$, the curves do not pass through the origin. Different parameters giving different curves may apply for different segments of a motion. For example, the two curves

$$\begin{aligned} f &= f(AB) = (1-\sqrt{2}\exp[-x]) \\ \bar{f} &= f(CD) = -(1-\sqrt{2}\exp[\bar{x}]) \end{aligned} \tag{7}$$

with $\omega^2=\bar{\omega}^2=1$, $\alpha=1=-\bar{\alpha}$ and $\delta=\bar{\delta}=-\sqrt{2}$ are seen to be discontinuous in Fig. 1. This restoring force will be used later in an example of an oscillation between the turning points $x=\pm.881$. Here barred values apply for motion in the negative x direction while unbarred values denote motion in the positive x direction. Another example to be described later uses the restoring force segments

$$
\begin{aligned}
f &= -1.703(1-.5\exp[x]) \\
f &= \ \ 1.703(1-.5\exp[-x])
\end{aligned}
\tag{8}
$$

with $\omega=1.305$, $\alpha=\pm1$ and $\delta=-.5$ which form the closed loop ab in Fig. 1. The first f is zero for $x=.693$ and describes the lower segment while the second f vanishes for $x=-.693$ and describes the upper segment. We have not used bars in Eq (8) to specify the direction of motion for each segment. A particular application however will require specification of clockwise or counterclockwise motion around the loop. As with any hysteresis loop, the integral of f around the loop does not vanish.

Let us describe how the loop in Fig 1 might be used to represent a cyclic stress-strain ($\sigma$-$\epsilon$) curve for example[4]. Let x represent a displacement which we multiply by $|\alpha|=(1/L)$ where L is a characteristic length. Then $\epsilon=|\alpha|x$ is a strain. Let f represent a force per unit mass which we multiply by $\rho L$ where $\rho$ is the density of the solid sample. Then $\sigma=\rho Lf$ is a stress. Note that the turning points of the loop depend only on $\delta$ and $\alpha$ not $\omega$. This may be seen from the condition of loop closure,namely, $(\omega^2/\alpha)[1+\delta\exp(-\alpha x_0)]$ $=-(\omega^2/\alpha)[1+\delta\exp(\alpha x_0)]$ when we return to the starting point $x_0$. This leads to $.5[\exp(\alpha x_0)+\exp(-\alpha x_0)] = \cosh(\alpha x_0) = -1/\delta$, so if $\delta=-.5$, then $\alpha x_0=\pm1.317$ as shown. Now suppose the sample we are using is initially free of stress and strain. That is, we start at the origin in Fig 1 instead of at some point on the loop. For $f=0$ when $x=0$ in Eq (6), we must have $\delta=-1$. Let $\rho L\omega^2=1$ for convenience and represent the first segment by ${}_1\sigma=[1-\exp(-\epsilon)]$, using $\alpha=1$ and $\epsilon$ increasing ($v=\dot{x}>0$). In practice the initial portion is a straight line and we should start our first arc segment where the elastic limit is reached. However, for simplicity we will use an arc from the beginning for purposes of illustration. In Fig 2 we see that as we increase the pull on the sample ($\sigma>0$ increasing) we increase $\epsilon$ to 1 when $\sigma=.632$ in the units we are using. Now suppose we begin to release the tension, decreasing $\epsilon$ ($v<0$). When the stress has been removed ($\sigma=0$) we find a residual elongation ($\epsilon>0$). The second arc segment is described by ${}_2\sigma = -[1-.6\exp(\epsilon)]$ where $\epsilon=|\alpha|x$ with $\alpha=-1$. The value $\delta=-.6$ was found from the continuity condition at the end of the first segment or beginning of the second segment namely, ${}_2\sigma = -[1+\delta\exp(1)] = .632$. To reduce the residual elongation to zero we must apply pressure ($\sigma<0$). If we continue to increase the pressure after the original length has

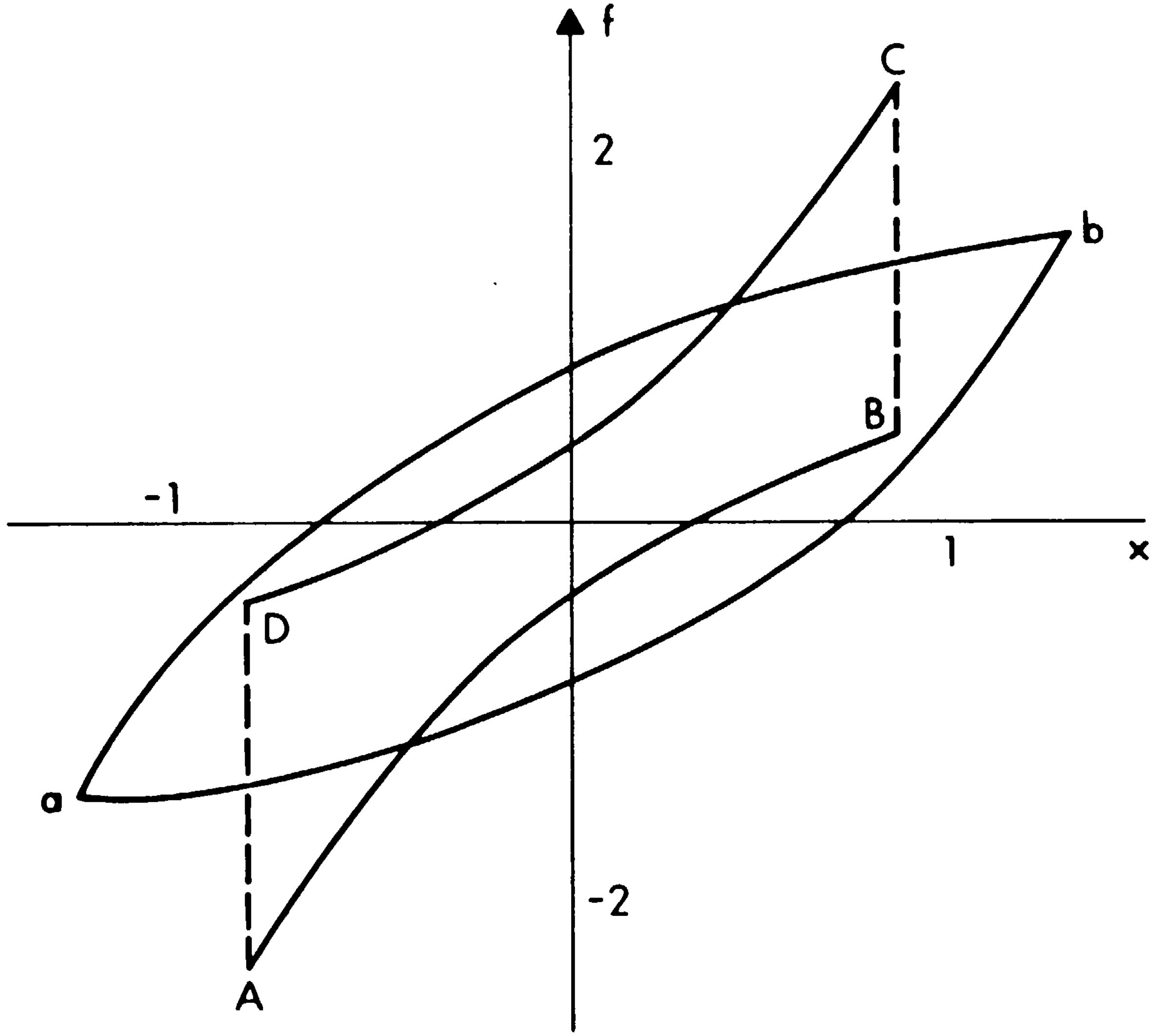


Figure 1. Examples of Two-Segment Restoring Forces.

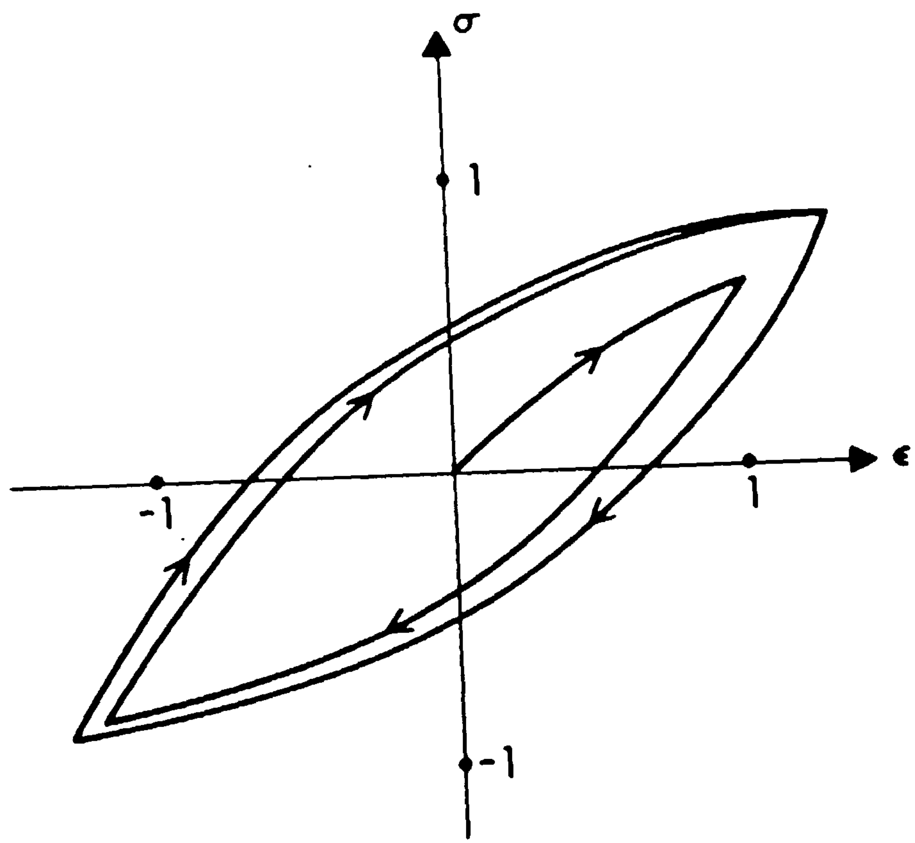


Figure 2. Approach to a Cyclic Stress-Strain Curve.

been restored, we compress the sample ($\epsilon<0$), and when ${}_2\sigma=-.819$, we find $\epsilon=-1.2$. Now we gradually release the pressure, allowing the sample length to grow ($v>0$). This time when the stress is removed we find a residual compression, so we must pull the sample ($\sigma>0$) to restore its original length ($\epsilon=0$). Continuity at the beginning of the third segment requires ${}_3\sigma=(1+\delta\exp[-(-1.2)])=-.819$ so ${}_3\sigma=[1-.548\exp(-\epsilon)]$. If we pull until ${}_3\sigma=.85$ and $\epsilon=1.3$ then start to release, we have ${}_4\sigma=-[1+\delta\exp(1.3)]=.851$ so ${}_4\sigma=-[1-.504\exp(\epsilon)]$ and so on. Clearly $\delta$ is approaching $-.5$. The final loop is traversed repeatedly and is described by the two segments $\sigma=[1-.5\exp(-\epsilon)]$ and $\sigma=-[1-.5\exp(\epsilon)]$. For a metal alloy sample the stress might be measured in hundreds of megapascals for microstrains in units of $10^{-3}$. Clearly we can shift the position of such loops as well as change their size and shape by changing the parameters. Rotations are also possible.

Consider another example described by the four segments

$$
\begin{aligned}
f &= -8[1-\exp(-2+2x)] \\
f &= 8[1-\exp(2-2x)] \\
f &= 8[1-\exp(-2-2x)] \\
f &= -8[1-\exp(2+2x)]
\end{aligned}
\tag{9}
$$

which have been plotted in Fig 3 and might be used to describe a major ferromagnetic hysteresis loop. In the idealization of Fig 3 the loop is not quite closed for finite x. In Eq (9) $\omega=4$ $\alpha=\pm2$ and $\delta=-\exp(\pm2)$.

As one may readily imagine, a motion may be divided into more than two segments, each with its own restoring force, so that a great variety of figures can be constructed by combining f-curves. In some cases we may require f and/or its derivatives with respect to x to be continuous at certain switching points in the motion where parameter changes occur. This will lead to relations between the parameters. If x and t represent physical variables, we may want x and its derivatives with respect to t to be continuous. If X also represents a physical variable instead of merely being a parameter linking x and t, we may impose similar requirements. In this report we will limit ourselves to motions with no more than two segments and will usually consider x and t (but not X) to represent physical variables.

If there is no damping or forcing in Eq (1) it describes a conservative system. In this case, multiplying by $dX=\dot{X}dt=Vdt$ and integrating leads to

$$E = .5(V^2+\omega^2X^2) = .5(V_0^2+\omega^2X_0^2) = .5[(\omega A\sin\omega t)^2+\omega^2(A\cos\omega t)^2] = .5(\omega A)^2 \tag{10}$$

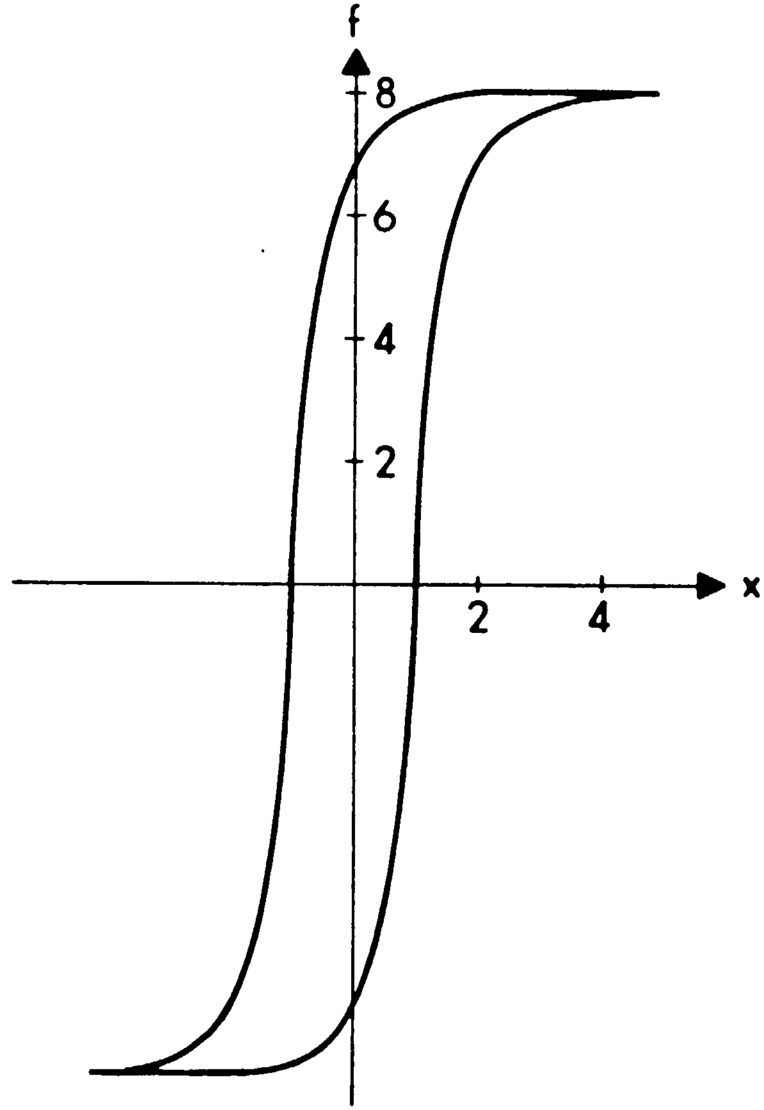


Figure 3. A Four-Segment Hysteresis Curve.

where E is the constant energy per unit mass in mechanical applications. Similarly in Eq (5) with $\beta=F=0$ so $\delta=\varsigma/\xi$, we can multiply by $\exp(2\alpha x)dx$ with $dx=\dot{x}dt$ and integrate to find

$$e = .5[(\dot{x}\exp[\alpha x])^2 + (\omega/\alpha)^2[(\exp[\alpha x]+\delta)^2 - \delta^2]]$$

$$= .5[(\dot{x}_0\exp[\alpha x_0])^2 + (\omega/\alpha)^2[(\exp[\alpha x_0]+\delta)^2 - \delta^2]] \qquad (11)$$

If we multiply Eq (11) by the dimensionless quantity $(\alpha\xi)^2$ we find

$$(\alpha\xi)^2 e = .5[V^2 + \omega^2(X^2-\varsigma^2)] = E - .5(\varsigma\omega)^2 \qquad (12)$$

using Eq (4) and its time derivative as well as Eq (10). Consequently, there can be an arbitrary constant energy in the nonlinear case even when $E=0$ in the linear case, provided neither $\varsigma$ nor $\omega$ but only A are zero. If X changes sign periodically in Eq (4) then $\varsigma$ cannot be zero for real x. For $\xi=1$ and $X=\cos(t)$ for example, we must have $\varsigma<-1$.

## II. THE LINEAR OSCILLATOR

The linear oscillator is often used to introduce many of the phenomena exhibited by nonlinear equations, since, in spite of obeying the principle of superposition, it can also have limit cycle solutions. Since the solutions of Eq (5) are built on those of Eq (1), it will be convenient to have some solutions of Eq(1) exhibited for reference.

It is common practice to discuss solutions in the X,V phase plane. If we let $V=\dot{X}$, Eq (1) becomes

$$dV/dX = -(\beta V+\omega^2 X-F_0)/V \qquad (13)$$

when $F=F_0$ is constant. Because of superposition we can write

$$X = X_c + X_p = X_c + \epsilon \qquad (14)$$

where $X_c(t)$ is the complementary solution for $F=0$ and $\epsilon=F_0/\omega^2$ is the particular solution which in this case merely shifts the origin. When $(\beta/2)^2\geq\omega^2$, the solutions are non-oscillatory with at most one turning point $(V=0)$. The phase plane trajectories are given by

$$[(V-\gamma^+X_c)/B^+]^{\gamma^+} = [(V-\gamma^-X_c)/B^-]^{\gamma^-} \qquad (15)$$

with

$$\gamma^{\pm} = -(\beta/2) \pm \sqrt{(\beta/2)^2-\omega^2} \tag{16}$$

and

$$B^{\pm} = V_0 - \gamma^{\pm} X_{co} \tag{17}$$

for initial values $V_0$ and $X_{co}=X_o-\epsilon$ by Eq (14) as shown, for example, by Minorsky[5]. In the $X_c$,t plane

$$X_c = [B^- e^{\gamma^+(t-t_0)} - B^+ e^{\gamma^-(t-t_0)}]/(\gamma^+-\gamma^-) \quad . \tag{18}$$

Special forms hold when $(\beta/2)^2=\omega^2$ so $\gamma^+=\gamma^-$ and $B^+=B^-$, as is well known.

When $(\beta/2)^2<\omega^2$, the solution is oscillatory and we have an infinite number of turning points. If we let $\omega_d=\sqrt{\omega^2-(\beta/2)^2}$ , Eq (16) becomes

$$\gamma^{\pm} = -(\beta/2) \pm i\omega_d \tag{19}$$

where $i=\sqrt{-1}$ and $\omega_d$ is the damped frequency. Eq (18) with $t_0=0$ becomes

$$X_c = A \exp(-\beta/2\ t) \cos(\omega_d t-\vartheta_0) \tag{20}$$

with

$$A = \pm\sqrt{X_{co}^2 + [(V_0+\beta X_{co}/2)/\omega_d]^2} \tag{21}$$

and

$$\vartheta_0 = \arctan [(V_0+\beta X_{co}/2)/(\omega_d X_{co})] \quad . \tag{22}$$

If we eliminate t between Eq (18) and its time derivative, we find for the trajectory in phase space

$$(V+\beta X_c/2)^2 + (\omega_d X_c)^2 = (\omega_d A)^2 \exp[-(\beta/\omega_d)(\vartheta_0-\vartheta)] \tag{23}$$

a logarithmic spiral. By convention we may let $\vartheta$ increase in a counterclockwise direction as t increases. The representative point in phase space then approaches or recedes from the origin, depending on the signs of $\beta$ and t.

A well-known particular solution occurs when F is sinusoidal. For example, if $F=F_0\exp[\pm iqt]$ where $q=2\pi\nu$ is the circular driving frequency,

$$X_p = (F_0/\Omega)\exp[\pm iqt] = (F_0/|\Omega|)[(\cos\phi \mp i\sin\phi)(\cos qt \pm i\sin qt)]$$
$$= (F_0/|\Omega|)[\cos(qt-\phi) \pm i\sin(qt-\phi)] = X_r \pm i X_i$$
$$= (F_0/|\Omega|)\exp[\pm i(qt-\phi)] \qquad (24)$$

with $$\phi = \arctan\,[q\beta/(\omega^2-q^2)] \qquad (25)$$

and $$\Omega = (\omega^2-q^2) \pm iq\beta \qquad (26)$$

with $|\Omega| = \sqrt{\Omega\Omega^*}$ where $\Omega^*$ is the complex conjugate of $\Omega$. If q is zero, $X_p \to F_0/\omega^2$ as before. By Eq (14) the general solution is the sum of Eqs (20) and (24). Since Eq (20) is a transient for $\beta>0$, the phase plane trajectory spirals toward the elliptical limit cycle

$$(qX)^2 + V^2 = (qX_0)^2 = 2E/m \qquad (27)$$

as $X \to X_p$ and the subscript p is omitted[6]. Here E is the limit cycle energy, while m is the mass or its analog in a non-mechanical application.

If F is a square wave in time instead of a sine wave such that $\epsilon = F_0/\omega^2$ changes sign but not magnitude at every turning point, we can have another example of a limit cycle[5], provided $\epsilon$ always aids the motion, countering the effect of $\beta>0$. Let us begin monitoring the motion at a turning point, $V_0=0$. Then by Eq (22), $\tan\vartheta_0 = \beta/(2\omega_d) = c$, defining c. By Eq (21), $A = X_{co}\sqrt{1+\tan^2\vartheta_0} = X_{co}/\cos\vartheta_0$ or, as in Eq (20), $X_{co} = A\cos\vartheta_0 = \kappa$, defining $\kappa$. Consequently, Eq (14) becomes

$$X_0 = \kappa + \epsilon \qquad . \qquad (28)$$

Since $\vartheta = \vartheta_0 - \omega_d t$, then when $\omega_d t_1 = \pi$, $\vartheta$ has decreased by $\pi$ and $\vartheta_0 - \vartheta = \pi$ in Eq (23), while $\cos(\omega_d t_1 - \vartheta_0) = -\cos\vartheta_0$ in Eq (20), then $X_{c1} = -A\cos\vartheta_0\,\exp(-c\pi) = -\kappa\,\exp(-c\pi)$ so

$$X_1 = \epsilon - \kappa e^{-c\pi} = -X_0 e^{-c\pi} + \epsilon(1+e^{-c\pi}) \qquad (29)$$

when we use Eq (28) to eliminate $\kappa$ in favor of $X_0$. This relates the end of the first segment to the beginning, $X_0$. For the second segment ($V\leq 0$), let us denote possible changes by using a bar over every quantity except t. At the beginning of this segment $t = t_1 = \pi/\bar{\omega}_d$, and by analogy with Eq (29)

$$\bar{X}_1 = \bar{\epsilon} - \bar{\kappa}\, e^{-\bar{c}\pi} = -X_0\, e^{-c\pi} + \epsilon(1+e^{-c\pi}) \tag{30}$$

where $\bar{c}=\bar{\beta}/(2\bar{\omega}_d)$. The second form of Eq (30) comes from requiring X to be continuous, namely $\bar{X}_1=X_1$. At the next turning point located on the opposite side of the origin from $\bar{X}_1$, $\theta$ has decreased by $2\pi$, so from Eq (20)

$$\begin{aligned}\bar{X}_2 &= \bar{\epsilon} + \bar{\kappa}\, e^{-2\bar{c}\pi} = \bar{\epsilon} - (\bar{X}_1-\bar{\epsilon})e^{-\bar{c}\pi} \\ &= \bar{\epsilon} - [-X_0 e^{-c\pi} + \epsilon(1+e^{-c\pi}) - \bar{\epsilon}]\, e^{-\bar{c}\pi} \\ &= \bar{\epsilon}(1+e^{-\bar{c}\pi}) - \epsilon\, e^{-\bar{c}\pi}(1+e^{-c\pi}) + X_0 \epsilon^{-(c+\bar{c})\pi} \quad .\end{aligned} \tag{31}$$

The second form of $\bar{X}_2$ is obtained by eliminating $\bar{\kappa}$ in favor of $\bar{X}_1$, using the first form of Eq (30). The third form of $\bar{X}_2$ is obtained by using the second form of Eq (29), while the final form is simply a rearrangement. If we now require $X_2=\epsilon+\kappa\,\exp[-2c\pi]=\bar{X}_2$, $X_3=\epsilon-\kappa\,\exp[-3c\pi]=\bar{X}_3$, and so on, we can relate each turning point $X_n$ to $X_0$. For even-numbered turning points we find

$$X_{2n} = X_0 e^{-n(c+\bar{c})\pi} + [\bar{\epsilon}(1+e^{-\bar{c}\pi}) - \epsilon\, e^{-\bar{c}\pi}(1+e^{-c\pi})]\, \Sigma_{k=0}^{n-1}\, e^{-k(c+\bar{c})\pi} \tag{32}$$

while for odd-numbered points

$$X_{2n+1} = -e^{-c\pi}X_{2n} + \epsilon(1+e^{-c\pi}) \tag{33}$$

where n=0,1,2.... As the number of turns increases without limit ($n\to\infty$), the first term in Eq (32) vanishes and the turning points become independent of the initial conditions, provided $(c+\bar{c})>0$. This illustrates another limit cycle solution of Eq (1). Recall

$$\lim_{n\to\infty} \Sigma_{k=0}^{n-1}\, e^{-k(c+\bar{c})\pi} = [1 - e^{-(c+\bar{c})\pi}]^{-1} \quad . \tag{34}$$

Eq (34) enables us to write the limit cycle turning points, Eqs (32) and (33), in terms of a finite number of elementary functions

$$X_L(\text{even}) = [\bar{\epsilon}(1+e^{-\bar{c}\pi}) - \epsilon e^{-\bar{c}\pi}(1+e^{-c\pi})]/[1-e^{-(c+\bar{c})\pi}] \tag{35}$$

$$X_L(\text{odd}) = [\epsilon(1+e^{-c\pi}) - \bar{\epsilon} e^{-c\pi}(1+e^{-\bar{c}\pi})]/[1-e^{-(c+\bar{c})\pi}] \quad . \tag{36}$$

Eqs (35) and (36) could have been found by requiring curve closure ($X_2=X_0$) or ($X_3=X_1$) as well as continuity. However, this would not bring out the fact that the limit cycle is approached no matter what the initial conditions may be. In addition, continuity and closure also hold for periodic trajectories like ellipses which are not limit cycles.

If $\bar{\epsilon}=\epsilon$ and $\bar{c}=c$, then $X_L$(even)$=X_L$(odd) and we have a spiral toward the vertex $\epsilon$ instead of a limit cycle. If $\bar{c}=c=0$ (no damping) and $\bar{\epsilon}=\epsilon$, Eqs (35) and (36) are indeterminate. The trajectory is an ellipse with center $\epsilon$ and not a limit cycle. If $\bar{c}=c=0$ and $\bar{\epsilon}\neq\epsilon$, no continuity is possible. The trajectory consists of discontinuous elliptical arcs and is not a limit cycle. If $\bar{\epsilon}=-\epsilon$ and $\bar{c}=c>0$,

$$X_L(\text{odd}) = -\,X_L(\text{even}) = \epsilon(1+e^{-c\pi})/(1-e^{-c\pi}) \tag{37}$$

and the limit cycle is symmetric about the origin.

The restoring force, $\omega^2 X$, in Eq (1) adds and subtracts equal amounts of energy each half-cycle and so is not a factor in determining the existence of a limit cycle, provided the motion is oscillatory. Fig 4 shows an example of a symmetric limit cycle with $-\bar{\epsilon}=\epsilon=1/\sqrt{3}=.577$, $\bar{\omega}=\omega=1.024$ and $\bar{\beta}=\beta=.440$, so $\bar{\omega}_d=\omega_d=1$, $\bar{c}=c=.220$ and $\theta_0=\arctan c = .217$. From Eq (37) we find $X_0=-\sqrt{3}=X_{co}+.577$ by Eq (14), so

$$X_{co} = -\sqrt{3} - .577 = -2.309 = A\cos(-.217) \tag{38}$$

from Eq (20) and $A=-2.364$ as from Eq (21). The time-dependent solution is given by Eq (14) together with Eq (20) and its time derivative. For $V>0$

$$X = -2.364e^{-.22t}\cos(t-.217) + .577 \tag{39}$$

$$V = 2.364e^{-.22t}[.22\cos(t-.217) + \sin(t-.217)] \tag{40}$$

At the first turning point $t_1=\pi/\omega_d=\pi$ and $\theta_1=\theta_0-\pi= -2.925$, so $\cos[(t-t_1)-(\theta_0-\pi)]=\cos(t-.217)$ once more. The damping factor becomes $\exp[-.22(t-\pi)]$ which is equal to unity at the beginning of the second segment when $t=t_1=\pi$. Then $X_1=\sqrt{3}=\bar{X}_1=\bar{A}\cos(2.925)-.577$, so $\bar{A}=A$. Half-cycles with $V<0$ are then described by

$$\bar{X} = -2.364\, e^{-.22(t-\pi)}\cos(t-.217) - .577 \tag{41}$$

$$\bar{V} = 2.364\, e^{-.22(t-\pi)}[.22\cos(t-.217) + \sin(t-.217)] \tag{42}$$

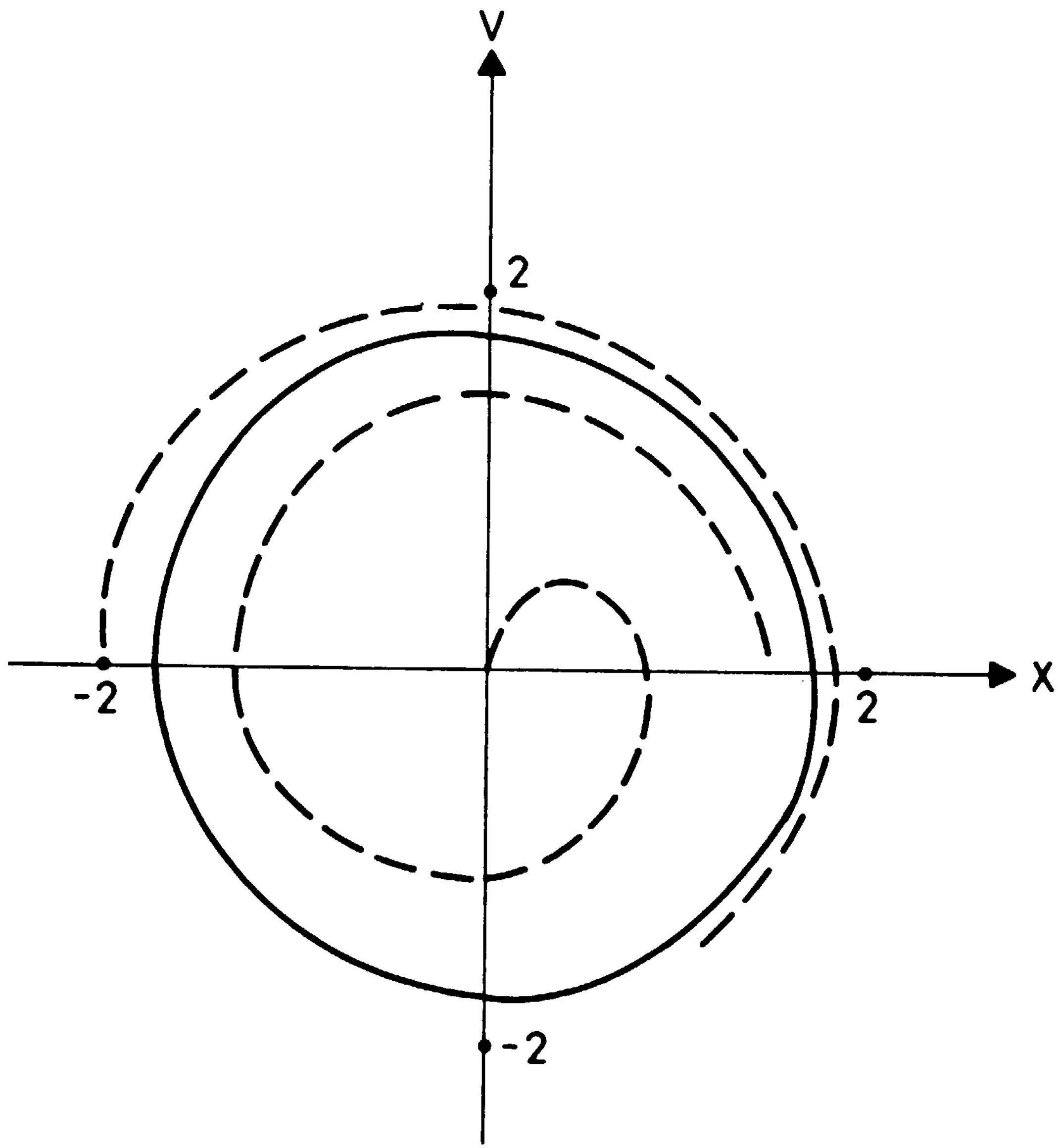


Figure 4. A Linear Limit Cycle.

Examples of other intial conditions inside or outside the limit cycle are also shown in Fig 4. If the trajectory starts from the origin ($X_0=V_0=0$), we find $X_{co}= 0 - .577 = A_o\cos(-.217)$ so $A_o=-.591$. Then for the first half-cycle with V>0, for $0<t<\pi$

$$X = -.591\ e^{-.22(t-0)}\cos(t-.217) + .577 \tag{43}$$

$$V = .591\ e^{-.22(t-0)}[.22\cos(t-.217) + \sin(t-.217)] \quad . \tag{44}$$

Since $\bar{X}_1=X_1=.866=\bar{A}_1\cos(2.925)-.577$, $\bar{A}_1=-1.478$ and

$$\bar{X} = -1.478\ e^{-.22(t-\pi)}\cos(t-.217) - .577 \tag{45}$$

$$\bar{V} = 1.478\ e^{-.22(t-\pi)}[.22\ \cos(t-.217) + \sin(t-.217)] \tag{46}$$

for $\pi<t<2\pi$. At $t_2$, $\bar{X}_2=-1.299=X_2=A_2\cos(6.066)+.577$, so $A_2=-1.921$. If we look at the sequence -.591,-1.478 and -1.921, we see that A is approaching the limit cycle value,-2.364, even as X and V are approaching their limit cycle forms given by Eqs (39)-(42). If the phase of the driving force were shifted initially by 180°, the motion would begin with X<0, V<0 and the trajectory is a reflection through the origin of the one shown. In either case, the system is self-excited and approaches a stable oscillation state even if started from rest at the origin.

If the trajectory starts from a point outside the limit cycle such as $X_0=-2$, $V_0=0$, we find $X_{co}=-2-.577=A_o\cos(-.217)$, so $A_o=-2.639$ and

$$X = -2.639\ e^{-.22(t-0)}\cos(t-.217) + .577 \tag{47}$$

$$V = 2.639\ e^{-.22(t-0)}[.22\ \cos(t-.217) + \sin(t-.217)] \tag{48}$$

for the first half-cycle with a turning point at $X_1=1.866=\bar{X}_1=\bar{A}_1\cos(2.925)-.577$, so $\bar{A}_1=-2.501$, beginning to approach the limit cycle value of -2.364. The procedure is clear, so there is no need to elaborate further on this example.

## III. CONSTANT PARAMETRIC EXCITATION

### A. Introduction

Once we specify $\beta,\omega^2$ and F, we can write solutions to Eq (1). If we also specify $\alpha,\xi$ and $\zeta$, we can use Eq (4) to write solutions to Eq (5). The number of possible combinations of six constant parameters is too large to

examine extensively in this report, so we will be highly selective. In this section we consider cases in which $F=F_0$, a constant (possibly zero).

When $F=\varsigma=0$, Eq (5) is separable. Its solutions are well-known and their application to penetration mechanics has been discussed[7].

B. Some non-oscillatory trajectories

Eq (15) reduces to a simple quadratic form in $X_c$ and V when $\gamma^+=-1$ and $\gamma^-=-2$, as occurs when $\omega^2=2$ and $\beta=3$ in Eq (16). When X and $X_c$ differ by a constant as in Eq (14), then the X,V trajectory segments are hyperbolic or parabolic depending on the characteristic which in turn depends on the intial conditions in Eq (17). Other values of $\gamma^+$ and $\gamma^-$ will give trajectories which are qualitatively similar.

In our first example we let $\omega^2=2$ and $\beta=3$ with $X_{co}=-1$ in Eq (17) so $B^+=V_0-1$ and $B^-=V_0-2$. A choice of $V_0$ in the interval $1<V_0<2$ will give $B^+$ and $B^-$ opposite signs so V, the derivative of Eq (18), will not vanish until $t\to\infty$. In other words, the X,V trajectory will have no turning point. Let us take $V_0=1.5$ in appropriate units. Then $B^+=.5=-B^-$ and the trajectory equations are for $\alpha=1$

$$X = -.5e^{-t}(1+e^{-t}) + F_0/2 = \xi e^{x} + \varsigma \tag{49}$$

$$V = e^{-t}(e^{-t} + .5) = \xi\ v\ e^{x} \tag{50}$$

Clearly V and v only vanish for $t\to\infty$ and $X\to F_0/2$. If $F_0\leq 0$, the X,V trajectory lies in the upper left quadrant of the X,V plane. If $F_0>0$, it may lie partially or entirely in the upper right quadrant. Other initial conditions can lead to trajectories in the lower half plane. From Eq (50) we see that v has the same or opposite sign as V depending on the sign of $\xi$. Suppose $\xi=1$ and $F_0=0$. Then by Eq (49), $\exp(x_0)=-1-\varsigma\geq 0$, so $\varsigma\leq -1$. Clearly, the initial value, $x_0$, as well as the final value can be negative, zero or positive. The X,V and x,v trajectories are easily sketched and will not be shown here. For other values of $\omega^2$ and $\beta$ the calculations are easier if we work with equations like Eqs (49) and (50) rather than with Eq (15) and its analog in the x,v plane obtained by substituting $X_c=[\xi\exp(\alpha x)+\varsigma]-\epsilon$ and $V=\alpha\xi v\exp(\alpha x)$ in Eq (15). For applications to impact attenuation, for example, with given initial conditions $(x_0,v_0)$, we may adjust the linear $(\beta)$, quadratic $(\alpha)$ and

Coulomb or dry friction ($F_0$) damping as well as the other parameters in the exponential stiffness in order to obtain a desired final value of x as $v \to 0$.

In our second example, let us consider a case in which the motion is not only stopped but partially reversed (a rebound in a mechanical system). If we let $X_{co}=0$, then $B^+=B^-=V_o$ in Eq (17). In this case as well as others in which the $V_0$ term dominates, $B^+$ and $B^-$ have the same sign so V as well as v will vanish after a finite time and a single turning point can occur in each of the X,V and x,v trajectories. Here we will consider some examples in which the turning points are also switching points at which the parameters can change. In our examples we will keep $\omega^2=2$ and $\beta=3$ with $F_0=0$ throughout the motion. If $X_0=0$, $V_0=1$ and $-\zeta=\xi=\alpha=1$, we find

$$X = e^{-t}(1-e^{-t}) = e^{x}-1 \tag{51}$$

$$V = e^{-t}(2e^{-t}-1) = v\, e^{x} \tag{52}$$

with the restoring force $f=2(1-\exp[-x])$. The first segment of the X,V trajectory lies in the upper right quadrant as does the first x,v segment, as shown in Fig 5. Turning points occur ($V=v=0$) when $t_1=\ln 2$ at $X_1=.25$ and $x_1=.223$. If none of the parameters change, then Eqs (51) and (52) describe the second segments as well. Both of these have minima followed by an approach to the origin as $t \to \infty$. These are shown by the solid curves in the lower half planes in Fig 5. Different choices for $\alpha,\xi$ and $\zeta$ could lead to trajectories less similar than shown in Fig 5. In our example the linear damping, the quadratic damping and the restoring force all oppose the motion for $t<t_1$. After $t_1$, keeping $-\zeta=\xi=\alpha=1$ changes the sense of the quadratic damping and the restoring force from opposing to aiding the motion, while the linear damping continues to oppose the motion.

Suppose changes occur in $\alpha,\xi$ and $\zeta$ at the turning point. If x represents a physical variable which must be continuous, the condition required is $\bar{x}_1=x_1$ or

$$\bar{x}_1 = (1/\bar{\alpha})\ \ln[(\bar{X}_1-\bar{\zeta})/\bar{\xi}] = x_1 = (1/\alpha)\ \ln[(X_1-\zeta)/\xi] \tag{53}$$

where a bar denotes a value after the turning point and the right subscripts denote values at $t_1$. It is possible to choose parameters which make both x and X continuous. However, if X does not represent a physical variable, then there is no need to do this. As an example let us take $\bar{\alpha}=-1$ so the quadratic damping opposes the motion in the second segment as well. The the restoring force might or might not change its sense, depending on our choice of $\bar{\zeta}/\bar{\xi}$.

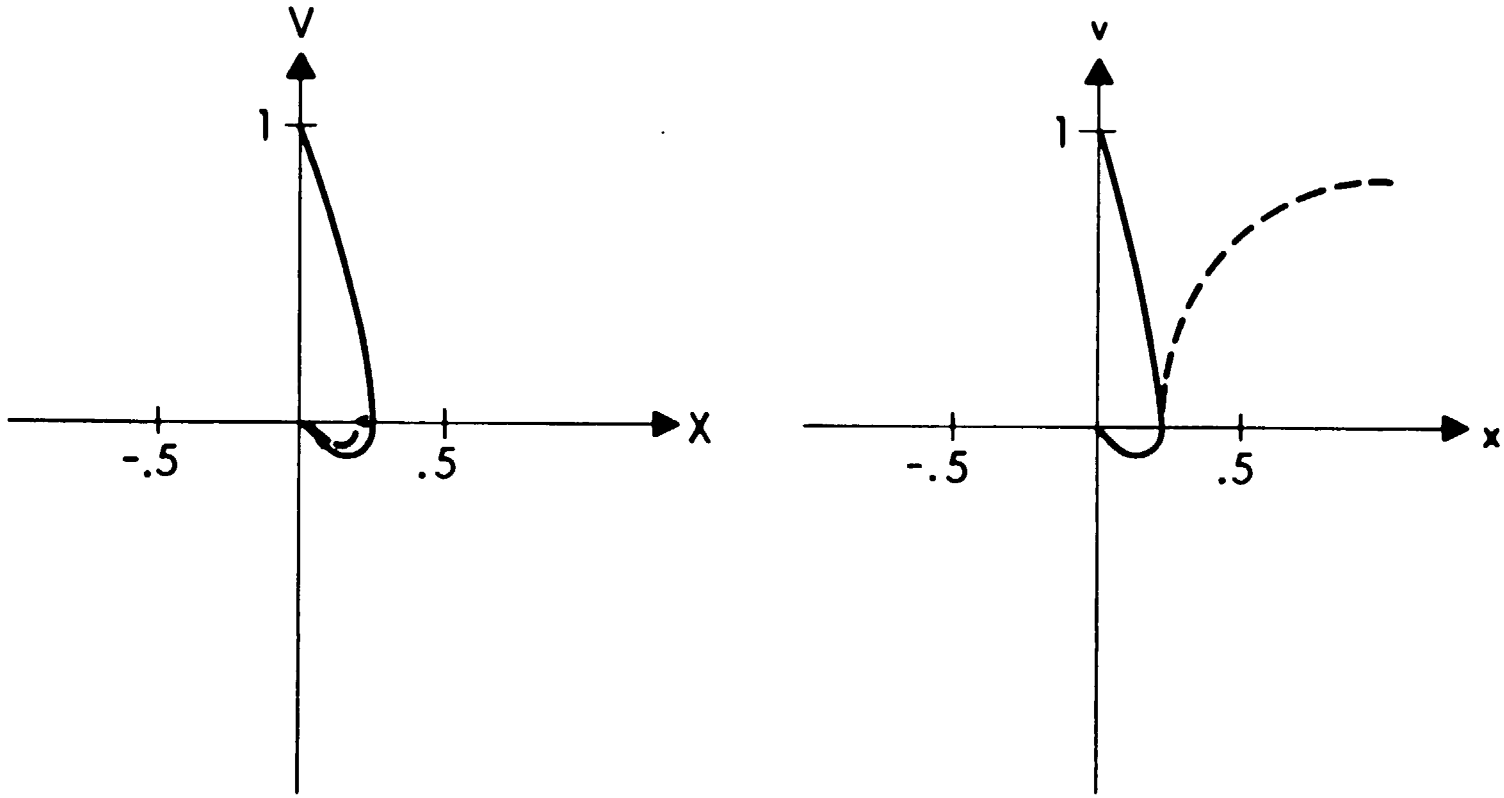


Figure 5. Rebound Trajectories.

Since $\bar{v}_1=0$ at the beginning of the second segment, $\bar{f}$ is the only force acting and its sense influences the shape of the second segment strongly, provided it is not initially zero. If $\bar{f}_1=0$, then the x-acceleration is momentarily zero. However, the motion continues because the X-acceleration is $-2\bar{X}_1\neq 0$ in our example.

Suppose $-\bar{\zeta}=\bar{\xi}=\bar{\alpha}=-1$ so $\bar{f}_1=-2[1-\exp(.223)]=.5$ and aids motion in the negative x direction. Let us also choose x continuous so $\bar{X}_1=X_1/(1+X_1)=.2$ by Eq (53). Since $\bar{V}_1=0$, we find $\bar{B}^{+}=.2$ and $\bar{B}^{-}=.4$. The initial time in Eq (18) is no longer zero but $t_1=\ln 2$. Then we find

$$\bar{X} = .8e^{-t}(1-e^{-t}) = 1 - e^{-\bar{x}} \tag{54}$$

$$\bar{V} = -.8e^{-t}(1-2e^{-t}) = \bar{v}\, e^{-\bar{x}} \tag{55}$$

for the second segments. The discontinuity in X is evident in the dashed X,V trajectory in Fig 5. The new $\bar{x},\bar{v}$ trajectory lies too close to the solid line to be distinguished in the scale of the figure.

Suppose we choose $\bar{\zeta}=0$. Then $\bar{f}=-2$, a constant opposing motion in the negative x direction. If we also choose $\bar{\xi}=.3125$, we (incidentally) keep X as well as x continuous. The trajectory equations are

$$\bar{X} = e^{-t}(1-e^{-t}) = .3125\, e^{-\bar{x}} \tag{56}$$

$$\bar{V} = -e^{-t}(1-2e^{-t}) = -.3125\, \bar{v}\, e^{-\bar{x}} \tag{57}$$

so the original solid line X,V trajectory has been restored as in Eqs (51) and (52). However, $\bar{v}>0$ when $\bar{V}<0$ by Eq (57) and the second segment of the x,v trajectory is now given by the dashed line in Fig 5. By dividing Eq (57) by Eq (56) we find

$$\bar{v} = (1-2e^{-t})/(1-e^{-t}) \tag{58}$$

for $t\geq \ln 2$ and $\bar{v}\to 1$ as $t\to\infty$ and $\bar{x}\to 0$.

If we include $F_0\neq 0$, more possibilities arise. For example, in the last case above, $\bar{v}>0$ can have a maximum followed by an approach to zero instead of an approach to 1 as in Eq (58). Or we can construct an x,v trajectory which terminates at a particular value of x after a finite instead of an infinite time. An example of the latter situation occurs if we add $F_0=-1=-\bar{F}_0$

(a dry friction force which always opposes the X motion). If $X_0=.366$ instead of zero and $V_0=1$ as before, we find

$$X = 2.732e^{-t} - 1.866e^{-2t} - .5 = e^{x} - 1 \qquad (59)$$

$$V = -2.732e^{-t} + 3.732e^{-2t} = v\, e^{x} \qquad (60)$$

with $-\zeta=\xi=\alpha=1$ as before and $x_0=.312$, $v_0=.732$ instead of 0,1. Now turning points occur when $\exp(-t_1)=.732$ at $X_1=.5$ and $x_1=.405$. If $\alpha,\zeta$ and $\xi$ do not change, then $\bar{X}_1=X_1=.5$ by Eq (53). But $\bar{F}_0=1$, so $\bar{X}_{c1}=\bar{X}_1-.5=0$. Since $\bar{V}_1=0$ also, we find $\bar{B}^{+}=\bar{B}^{-}=0$ from Eq (13). This implies that X remains at .5 and V remains zero with $x=.405$, $v=0$ for all future time.

These examples should suffice to indicate some of the types of behavior which are possible.

## C. Some Oscillatory Trajectories

### 1. Elementary Examples

First of all, suppose $\beta=0$, $\omega^2=4$ and $F_0=0$ with $X_0=x_0=0$ and $V_0=v_0=1$. In addition, let $-\zeta=\xi=1/\alpha=1$, so

$$X = .5 \sin 2t = e^{x} - 1 \qquad (61)$$

$$V = \cos 2t = v\, e^{x} \quad . \qquad (62)$$

The X,V trajectory segment is an elliptical arc, while the x,v segment is similar in appearance. Both are shown by solid lines in the upper right quadrants of Fig 6. The first turning points occur at $X_1=.5$ and $x_1=.405$ when $t_1=\pi/4$. If we require x to be continuous, we find $\bar{X}_1=1/3\neq X_1$, so X is discontinuous. The second segments for negative velocities are then described by using $\cos[2(t-\pi/4)]=\sin 2t$ so

$$\bar{X} = 1/3 \sin 2t = 1 - e^{-\bar{x}} \qquad (63)$$

$$\bar{V} = 2/3 \cos 2t = \bar{v}\, e^{-\bar{x}} \qquad (64)$$

if $\bar{\beta}=\beta=0$, $\bar{\omega}=\omega=2$, $-\bar{\zeta}=\bar{\xi}=1/\bar{\alpha}=-1$. The second turning points are $\bar{X}_2=-1/3$ and $\bar{x}_2=-.288$ when $t=3\pi/4$. To keep x continuous, we find $X_2=-.25$, so

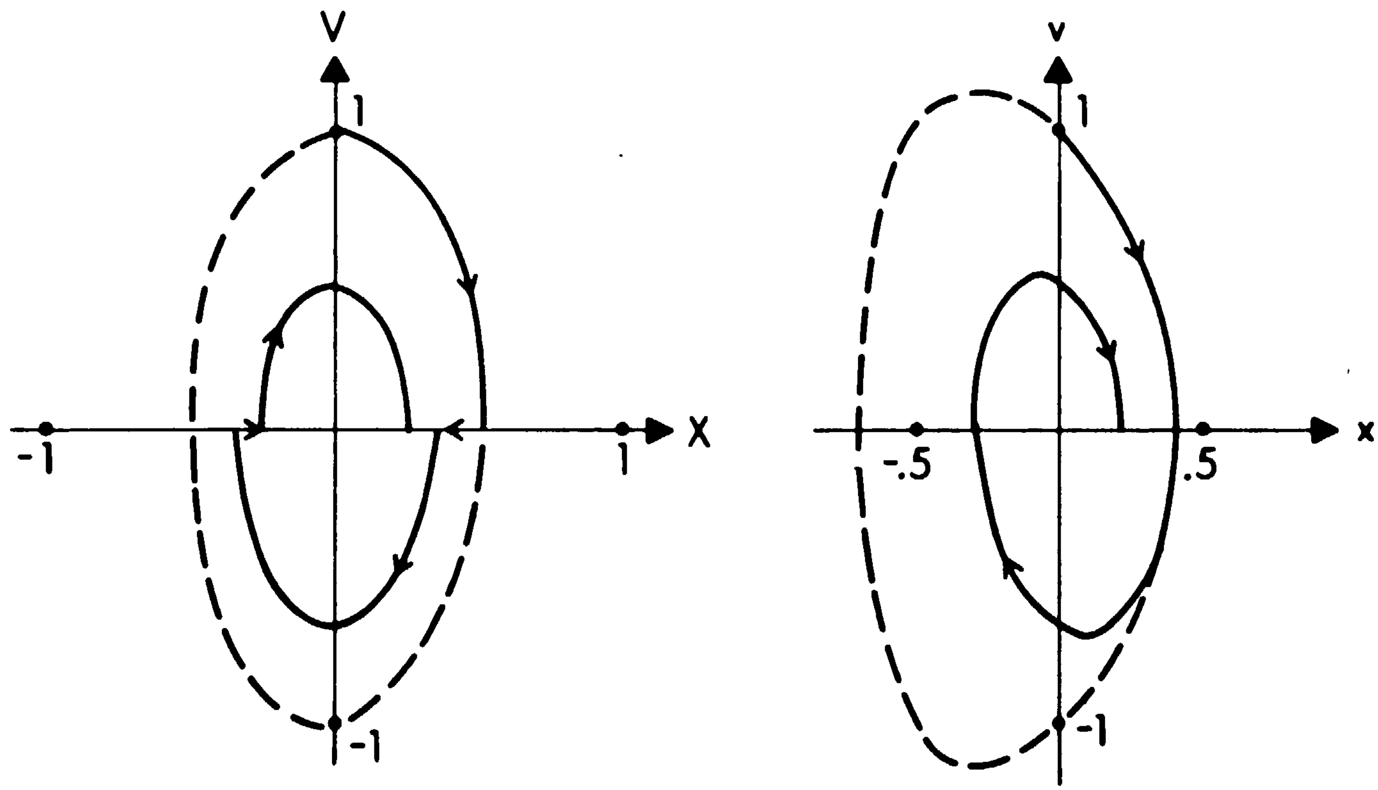


Figure 6. Examples of Oscillatory Trajectories.

$$X = .25\ \sin 2t = e^{x} - 1 \tag{65}$$

$$V = .5\ \cos 2t = v\ e^{x} \tag{66}$$

describe the third segments, and so on. It is clear from Fig 6 that the X,V trajectory consists of a series of discontinuous elliptical arcs with decreasing amplitudes 1/2, 1/3, 1/4, etc., while the x,v trajectory is a spiral approaching the origin. If $\alpha$ did not change sign at the turning points, Eqs (61) and (62) would describe the entire motion and we would obtain the closed trajectories shown by the dashed lines continuing the first segments. The X,V trajectory would be an ellipse. The alternation of positive and negative quadratic damping would always bring the nonlinear x,v oscillator back to its original energy state. In neither case do we have a limit cycle.

If we allow positive or negative linear damping we obtain spiral X,V arcs instead of elliptical arcs, while the x,v trajectory will spiral more rapidly or more slowly, depending on the sense of the linear damping. If we also allow nonzero $F_0$, other possibilities arise as one can readily imagine.

2. Two-segment Limit Cycles

a. Introduction

We will limit ourselves to cases in which parameter changes occur only at turning points with a return to the original parameter set each time $\vartheta$ changes by $2\pi$. We will also keep x continuous.

From Eqs (4) and (29)

$$\begin{aligned}\exp(\alpha x_1) &= \xi^{-1}(X_1-\zeta) = \xi^{-1}[-X_0e^{-c\pi}+\epsilon(1+e^{-c\pi})-\zeta] \\ &= \xi^{-1}(\epsilon-\zeta)(1+e^{-c\pi})-e^{-c\pi}[\xi^{-1}(X_0-\zeta)] \\ &= b_2-e^{-c\pi}\exp(\alpha x_0)\end{aligned} \tag{67}$$

where $b_2=\xi^{-1}(\epsilon-\zeta)(1+e^{-c\pi}) = -\delta(1+e^{-c\pi})$. Eq (67) relates the first turning point, $x_1$, to the initial turning point, $x_0$. Continuity of x requires $\bar{x}_1=x_1$, so from Eq (53)

$$\bar{X}_1 = \bar{\zeta}+\bar{\xi}[\xi^{-1}(X_1-\zeta)]^{\bar{\alpha}/\alpha} = \bar{\zeta}+\bar{\xi}\exp(\bar{\alpha}x_1) \tag{68}$$

using Eq (4) again. From the second form of Eq (31)

$$\bar{X}_2 = \bar{\epsilon}(1+e^{-\bar{c}\pi}) - \bar{X}_1 e^{-\bar{c}\pi} \quad . \tag{69}$$

Continuity of x requires $\bar{x}_2 = x_2$, so from Eq (53)

$$\xi^{-1}(X_2-\zeta) = [(\bar{\xi})^{-1}(\bar{X}_2-\bar{\zeta})]^{\alpha/\bar{\alpha}} \quad . \tag{70}$$

The left side of Eq (70) is $\exp(\alpha x_2)$ by Eq (4). From Eqs (67) to (70)

$$\begin{aligned}\exp(\alpha x_2) &= [\ \bar{\xi}^{-1}(\bar{\epsilon}-\bar{\zeta})(1+e^{-\bar{c}\pi}) - e^{-\bar{c}\pi}\exp(\bar{\alpha}x_1)\ ]^{\alpha/\bar{\alpha}} \\ &= [\ b_1 - e^{-\bar{c}\pi}[b_2 - e^{-c\pi}\exp(\alpha x_0)]^{\bar{\alpha}/\alpha}\ ]^{\alpha/\bar{\alpha}}\end{aligned} \tag{71}$$

where $b_1 = \bar{\xi}^{-1}(\bar{\epsilon}-\bar{\zeta})(1+e^{-\bar{c}\pi}) = -\bar{\delta}(1+e^{-\bar{c}\pi})$. Eq (71) relates the second turning point, $x_2$, to the intial value, $x_0$.

By analogy with Eq (67)

$$\begin{aligned}\exp(\alpha x_3) &= b_2 - e^{-c\pi}\exp(\alpha x_2) \\ &= b_2 - e^{-c\pi}[b_1 - e^{-\bar{c}\pi}[b_2 - e^{-c\pi}\exp(\alpha x_0)]^{\bar{\alpha}/\alpha}]^{\alpha/\bar{\alpha}}\end{aligned} \tag{72}$$

by using the second form of Eq (71). By analogy with the first form of Eq (71)

$$\begin{aligned}\exp(\alpha x_4) &= [b_1 - e^{-\bar{c}\pi}\exp(\bar{\alpha}x_3)]^{\alpha/\bar{\alpha}} \\ &= [b_1 - e^{-\bar{c}\pi}(b_2 - e^{-c\pi}[b_1 - e^{-\bar{c}\pi}[b_2 - e^{-c\pi}\exp(\alpha x_0)]^{\bar{\alpha}/\alpha}]^{\alpha/\bar{\alpha}})^{\bar{\alpha}/\alpha}]^{\alpha/\bar{\alpha}}\end{aligned} \tag{73}$$

using the $\bar{\alpha}/\alpha$ power of Eq (72), and so on.

We can write general expressions for even and odd turning points if we let O and $\bar{O}$ operating on a function Z be defined by

$$OZ = b_2 - e^{-c\pi}Z^{\alpha/\bar{\alpha}} \tag{74}$$

and

$$\bar{O}Z = b_1 - e^{-\bar{c}\pi}Z^{\bar{\alpha}/\alpha} \quad . \tag{75}$$

Then for the even-numbered turning points

$$\exp(\bar{\alpha}x_{2n}) = (\bar{O}O)^n \exp(\bar{\alpha}x_0) \tag{76}$$

and for the odd-numbered turning points

$$\exp(\alpha x_{2n+1}) = O\exp(\bar{\alpha} x_{2n}) \qquad (77)$$

for $n=0,1,2\ldots.$ The first value of n in Eq (76) gives an identity, while the next two values give the $(\bar{\alpha}/\alpha)$ powers of Eqs (71) and (73). The first two values of n in Eq (77) give Eqs (67) and (72). The form of equations like Eq (73) reminds one of a nest of dolls. Since $\alpha/\bar{\alpha}$ can have any value, these forms might also be called generalized continued fractions.

b. <u>Alternating Positive and Negative Quadratic Damping</u>

In the special case $\bar{\alpha}/\alpha = 1$, the quadratic damping is constant in direction and so alternates its sense from helping to hindering the motion and back every cycle. Eqs (76) and (77) become

$$\exp(\alpha x_{2n}) = e^{-n(c+\bar{c})\pi}\exp(\alpha x_0) + (b_1+b_2a_2)\,\Sigma_{k=0}^{n-1} e^{-k(c+\bar{c})\pi} \qquad (78)$$

and

$$\exp(\alpha x_{2n+1}) = b_2 + a_1\exp(\alpha x_{2n}) \qquad (79)$$

where we have introduced $a_1=-\exp(-c\pi)$ and $a_2=-\exp(-\bar{c}\pi)$. Eqs (78) and (79) are the nonlinear analogs of Eqs (32) and (33). As $n\to\infty$ both equations become independent of the initial condition, $x_0$, and we have expressions for the even and odd turning points of a limit cycle, provided $(c+\bar{c})>0$. From Eq (34)

$$\exp(\alpha x_{oL}) = (b_1+a_2b_2)/(1-a_1a_2) \qquad (80)$$

and

$$\exp(\alpha x_{1L}) = (b_2+a_1b_1)/(1-a_1a_2) \qquad (81)$$

where $x_{oL}$ is the value of all the even-numbered turning points of the limit cycle and $x_{1L}$ is the value of all the odd-numbered turning points. Eqs (80) and (81) are the analogs of Eqs (35) and (36). If the right sides of Eqs (80) and (81) are negative, we need complex $(\alpha x)$. Similar considerations will arise in the next subsection where we will give an example.

c. <u>Either Positive or Negative Quadratic Damping</u>

In some applications quadratic damping might have one magnitude and sense (either always aiding or always opposing the motion). This implies the special case $\alpha/\bar{\alpha} = -1$, so that Eqs (76) and (77) become continued fractions. For example, we can write Eq (72) as

$$\exp(\alpha x_3)-b_2 = \cfrac{a_1}{b_1+\cfrac{a_2}{\exp(\alpha x_1)}} \tag{82}$$

using Eq (67), relating the second odd turning point, $x_3$, to the first odd turning point, $x_1$. Similarly,

$$\exp(\alpha x_5)-b_2 = \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_1}{b_1+\cfrac{a_2}{\exp(\alpha x_1)}}}} \tag{83}$$

and so on. From Eqs (82) and (83) we see that we are dealing with a periodic continued fraction with period two. Our notation has been chosen to facilitate comparison with Wall's theorem[8]. Here we will let z represent the limit of the series of fractions begun in Eqs (82) and (83). Wall's constants $A_i$ and $B_i$ are related to our constants as follows: $A_1=a_1$, $B_1=b_1$, $A_2=b_2a_1$ and $B_2=b_2b_1+a_2$ with $A_{-1}=B_0=1$ and $B_{-1}=A_0=b_0=0$. Then the value of the fraction (the limit of the series) is one root of the quadratic equation

$$b_1z^2 + (b_2b_1+a_2-a_1)z - b_2a_1 = 0 \tag{84}$$

In other words, $z = \lim_{n\to\infty} (\exp[\alpha x_{2n+1}]-b_2) = \exp[\alpha x_{1L}] - b_2$, or

$$\exp[\alpha x_{1L}] = [\ (b_1b_2-a_2+a_1) \pm \sqrt{(b_1b_2+a_1-a_2)^2 + 4a_2b_1b_2}\ ]/(2b_1) \tag{85}$$

Since $\exp[\alpha x_{1L}] = b_2 + a_1\exp[\alpha x_{oL}]$ by Eq (67), we find

$$\exp[\alpha x_{oL}] = [(-b_1b_2-a_2+a_1) \pm \sqrt{(b_1b_2+a_1-a_2)^2 + 4a_2b_1b_2}]/(2a_1b_1) \tag{86}$$

Consequently, the limit cycle odd and even turning points represented by $x_{1L}$ and $x_{0L}$, are independent of the initial conditions and depend only on the parameters. We could obtain the same relations by requiring curve closure, $x_2=x_0$, in Eq (71) or $x_3=x_1$ in Eq (72) or (82). Normally, positive or negative quadratic damping throughout a motion would lead to spirals toward or away from a fixed point. Eqs (85) and (86) give the parameter relations which are needed to prevent either of these possibilities from happening.

From Eqs (85) and (86) we see that linear damping is not required for an x,v limit cycle in contrast with the requirements for an X,V limit cycle. Even if $\bar{\beta}=\beta=0$ so $\bar{c}=c=0$ and $a_1=a_2=-1$, Eqs (85) and (86) are still determinate. Positive or negative linear damping will merely decrease or increase the size of an x,v limit cycle, provided oscillations are still allowed. Consequently, we can concentrate on examples which have no linear damping without much loss in generality. In such a case Eqs (85) and (86) simplify to

$$\exp[\alpha x_{0L}] = -\delta + \sqrt{\delta^2 - (\delta/\bar{\delta})} \tag{87}$$

$$\exp[\alpha x_{1L}] = -\delta - \sqrt{\delta^2 - (\delta/\bar{\delta})} \quad . \tag{88}$$

If we also require symmetry about the origin, namely $x_{1L}=-x_{0L}$, these equations require $\bar{\delta}=\delta<0$ with $\delta^2\geq1$ for real variables and parameters. If $\bar{\delta}<\delta<0$, the right sides of Eqs (87) and (88) will be real, positive and specify the turning points of a real limit cycle. Clearly there are many other possibilities which can make the right sides of these equations negative or complex and require complex $(\alpha x)$ or complex x for real $\alpha$.

i. Real x

First, let us build an x,v limit cycle on the X,V limit cycle of Fig 4 (although an X,V limit cycle is not required). We recall that $\bar{\beta}=\beta=.44$ and $\bar{\omega}=\omega=1.024$ so $\bar{\omega}_d=\omega_d=1$, $\bar{c}=c=.22$ and $\theta_0=.217$ with $-\bar{\epsilon}=\epsilon=.577$, giving Eqs (39) to (42). Let us choose $-\bar{\alpha}=\alpha=1$, $-\bar{\xi}=\xi=1$ and $-\bar{\zeta}=\zeta=-2$. Then $a_2=a_1=-\exp[-.22\pi]=-.5$, $\bar{\delta}=\delta=-2.577$ and $b_2=b_1=2.577(1.5)=3.866$. Then Eqs (85) and (86) with positive square roots give $x_{0L}=-1.317$ and $x_{1L}=1.317$. If we used the negative square roots we would obtain $\pm2$ which turn out to be spurious (not the value of the continued fraction). The trajectory equations become

$$X = -2.364\exp(-.22t)\cos(t-.217) + .577 = \exp(x) - 2 \tag{89}$$

$$V = 2.364\exp(-.22t)[.22\cos(t-.217)+\sin(t-.217)] = v\exp(x) \tag{90}$$

using Eqs (39) and (40) together with the nonlinear forms of our present example for the first half cycle. From Eq (89) we find that $x_{0L}=-1.317$ and not 2, since $X_{0L}=-1.732$. The segments with negative velocities are described by

$$\bar{X} = -2.364\exp[-.22(t-\pi)]\cos(t-.217) - .577 = 2 - \exp[-\bar{x}] \qquad (91)$$

$$\bar{V} = 2.364\exp[-.22(t-\pi)][.22\cos(t-.217)+\sin(t-.217)] = \bar{v}\exp[-\bar{x}] \qquad (92)$$

The limit cycles are shown in Fig 7. The X,V curve is the same as the solid curve in Fig 4 where we illustrated the point that all trajectories approach the limit cycle no matter what the initial conditions. Obviously the same is true of the x,v limit cycle.

Second, consider an example without linear damping or driving force, so $\bar{\beta}=\beta=\bar{\epsilon}=\epsilon=0$ and there can be no X,V limit cycle. If $-\bar{\alpha}=\alpha=1=\bar{\omega}=\omega$ with $-\bar{\xi}=\xi=1$ and $-\bar{\zeta}=\zeta=-\sqrt{2}$, then $\bar{\delta}=\delta=-\sqrt{2}$. These values in Eqs (87) and (88) give $x_{0L}=.881=-x_{1L}$, using the signs of the square roots shown. We can reverse the signs of the square roots and still have a single value for the continued fraction. Therefore, let us use $x_{0L}=-.881=-x_{1L}$ instead. The restoring force is given by Eq (7) and is illustrated in Fig 1. As the motion proceeds from $x=-.881$ to $x=.881$, the force changes from point A to point B. At first it is negative and aids motion in the positive x direction. Then it changes sign and joins the damping in opposing the motion. At the turning point it is discontinuous and jumps to point C. On the return trip it first aids the motion in the negative x direction, then changes sign, and so on.

The trajectory segments with positive velocity are described by

$$X = -\cos(t) = \exp[x] - \sqrt{2} \qquad (93)$$

$$V = \sin(t) = v\exp[x] \qquad (94)$$

while those with negative velocity are described by

$$\bar{X} = \cos(t) = \sqrt{2} - \exp[-\bar{x}] \qquad (95)$$

$$\bar{V} = -\sin(t) = \bar{v}\exp[-\bar{x}] \qquad (96)$$

The X,V trajectory is the unit circle while the x,v limit cycle is the closed curve shown in Fig 8. The dashed trajectories illustrate how other trajectories are attracted to the limit cycle. Corresponding dashed semicircles are also shown.

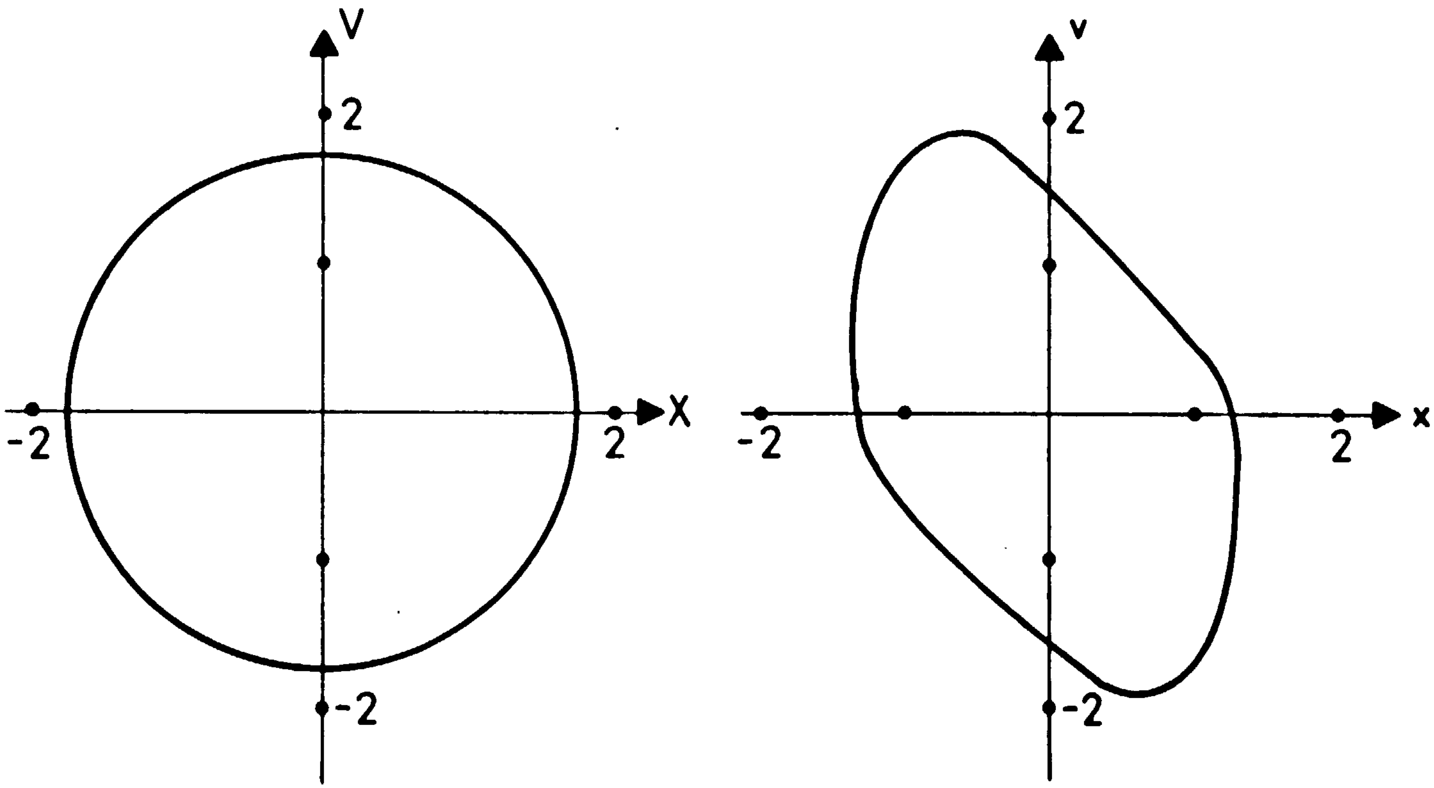


Figure 7. Nonlinear and Linear Limit Cycles.

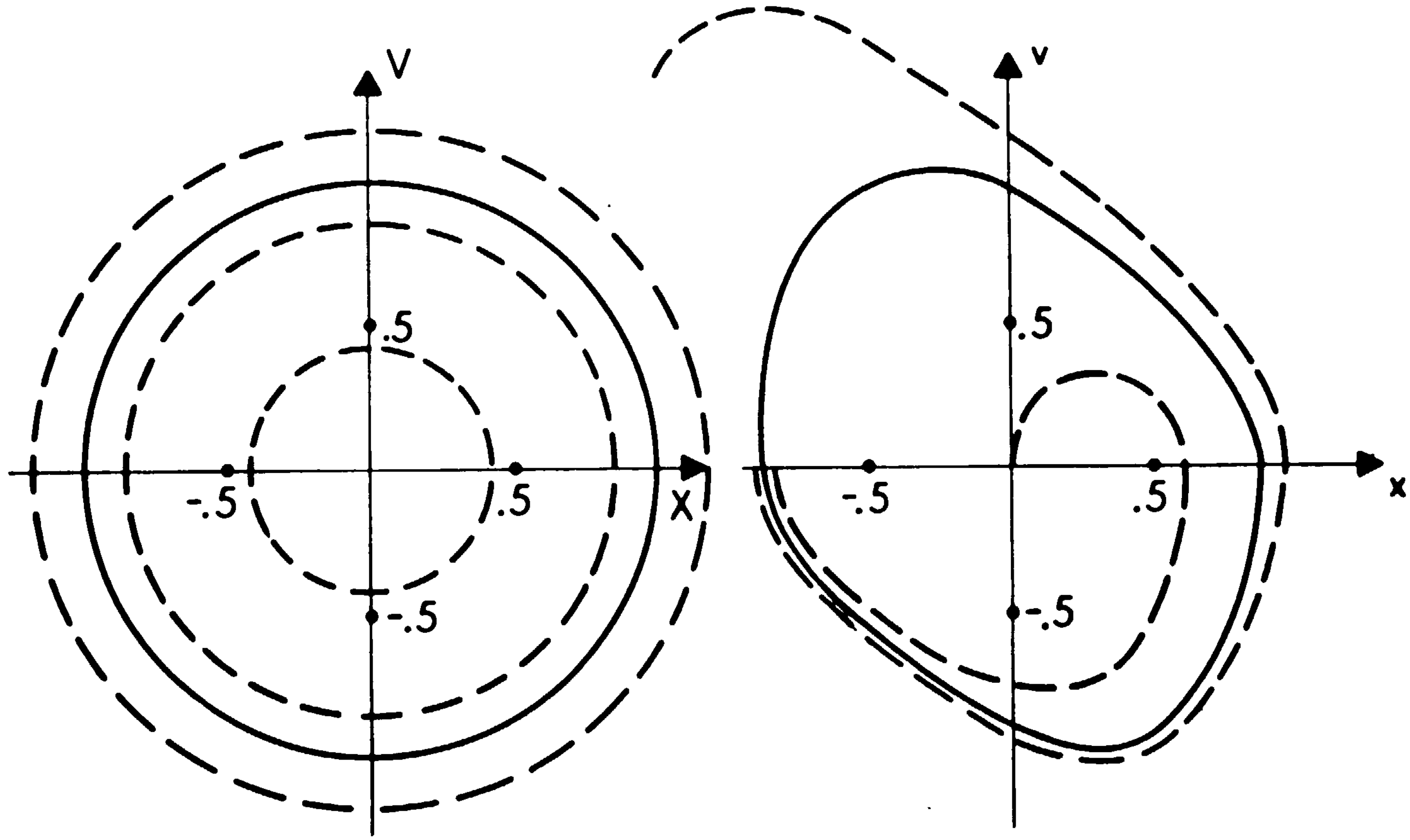


Figure 8. Nonlinear Limit Cycle and Linear Circle.

Suppose $-\bar{\zeta}=\zeta=-\sqrt{26}$ instead of $-\sqrt{2}$, while all the other parameters remain the same as for Fig 8. Then $\bar{\delta}=\delta=-\sqrt{26}$ and the equations are

$$X = -5\cos(t) = \exp[x] - \sqrt{26} \tag{97}$$

$$V = 5\sin(t) = v\exp[x] \tag{98}$$

and

$$\bar{X} = 5\cos(t) = \sqrt{26} - \exp[-\bar{x}] \tag{99}$$

$$\bar{V} = -5\sin(t) = \bar{v}\exp[-\bar{x}] \quad . \tag{100}$$

The X,V trajectory is a circle with radius 5, while the x,v limit cycle is shown in Fig 9. In Fig 10 we show v,t curves for $\delta=-\sqrt{2}$ and $-\sqrt{26}$. Such curves were named relaxation oscillations by van der Pol[9].

Finally, let us consider a limit cycle which involves the two-segment restoring force described by Eq (8), the loop in Fig 1. As we noted before, $\bar{\delta}=\delta$ must have a value between 0 and -1 for symmetric f arcs concave toward the origin. Consequently, $\delta^2<1$ and we cannot use Eqs (87) and (88) for real $(\alpha x_L)$. However, if $\bar{\beta}=\beta$ is not zero we can use Eqs (85) and (86). Since $a_1=a_2=a=-\exp(-c\pi)$ and $\bar{\delta}=\delta=-.5$, we have $b_2=b_1=b=.5(1-a)$. Then $c\geq.838$ or $c\leq-.838$ will lead to real $(\alpha x_L)$. If $c=.838$, $(-a)=.0718$, $(b/2)=.268$ and $(\alpha x_L)=\mp1.317$. If $c=-.838$, then $(-a)=13.91$, $(b/2)=3.728$ and $(\alpha x_L)=\pm1.317$. For either equality the square roots vanish in Eqs (85) and (86). Any other value of c gives turning points different from those shown in Fig 1, which (as we recall) were determined by our choice of $\delta$ and $\alpha$.

If we start at $x_L= -1.317$, then $v>0$ for the first segment and $\alpha=1$ (positive quadratic damping) means that we follow the upper arc in Fig 1 (the second form in Eq (8) ) and we move clockwise around the loop. Since $\omega^2=1.703$, let us choose $\omega_d=1$ so $c=\beta/2= \pm.838$ and $\theta_0=\arctan(c)=\pm.697$ for $V_0=0$. In addition, let us choose $F=0$ so $\delta=\zeta/\xi=-.5$. Then for positive linear damping we find $X_0=A(.767)=-.232\xi$ for $x_0=-1.317$ and at $t=\pi$ we find $X_1=A(-.767)(.0718)=3.232\xi$ for $x_1=1.317$. Of course there is no $A/\xi$ consistent with these requirements. However, if $c= -.838$ (negative linear damping), then $X_0$ remains the same while $X_1=A(-.767)(13.91)=3.232\xi$ and $A/\xi= -.303$ is consistent with both results. More generally, we find that for quadratic damping of one sense we must have linear damping of the opposite sense in order to move clockwise about the loop of Fig 1.

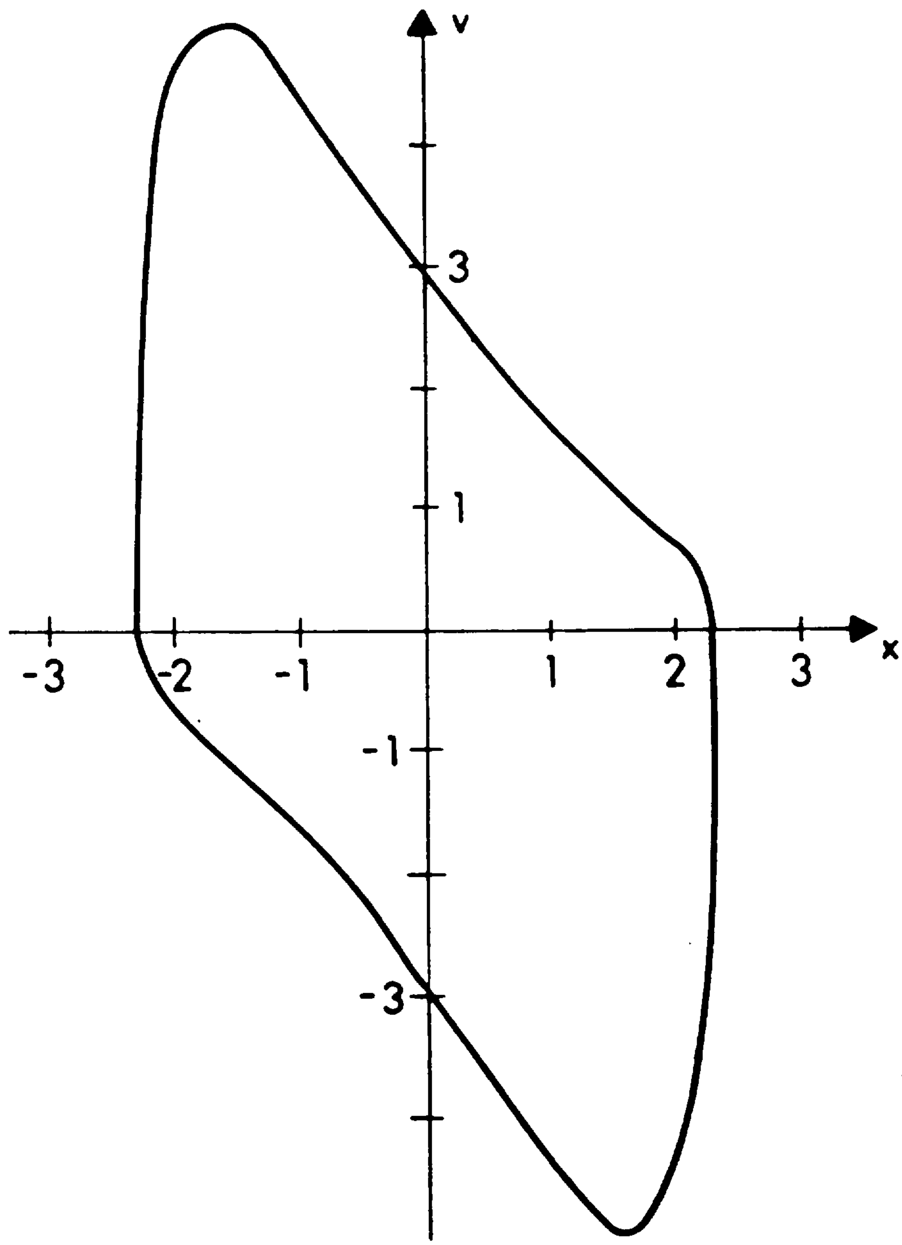


Figure 9. Limit Cycle for a Relaxation Oscillation.

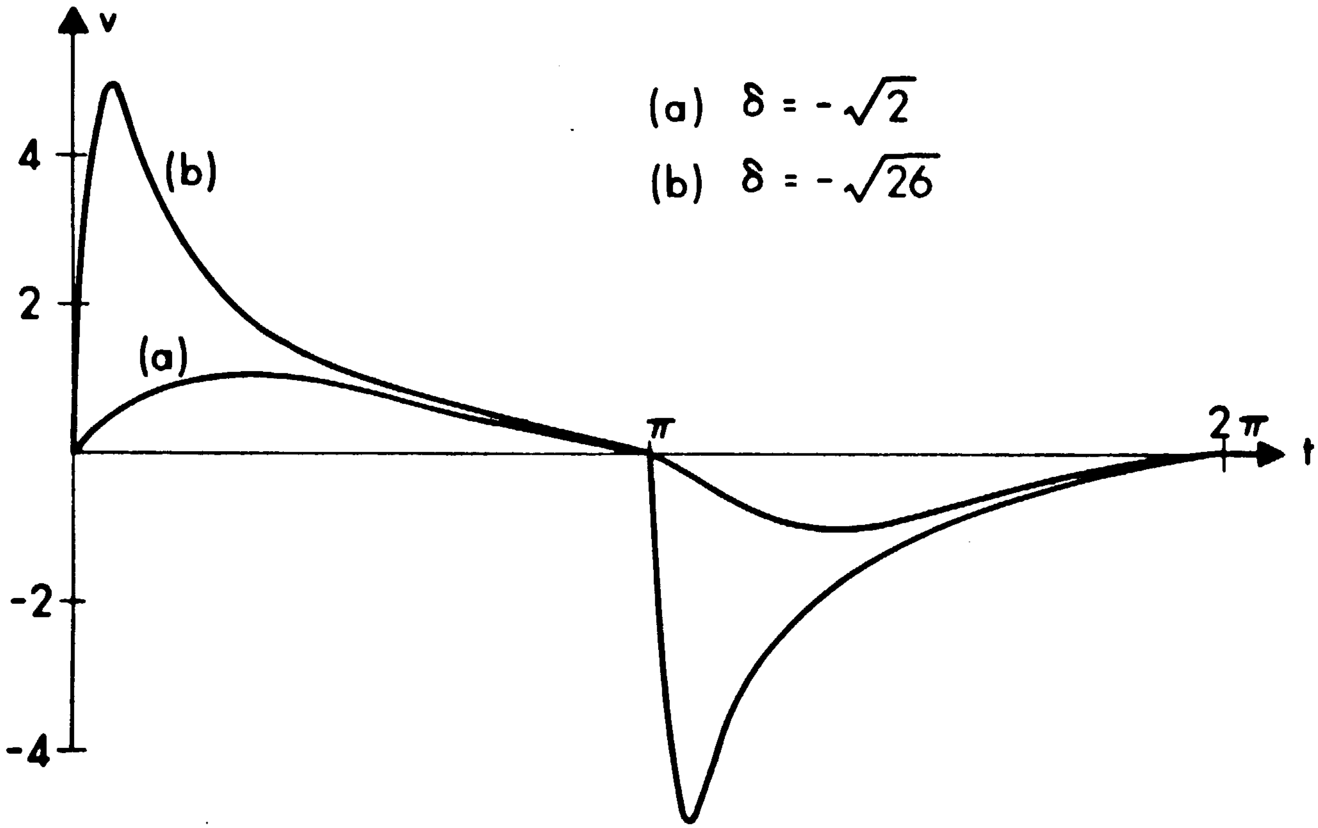


Figure 10. Evolution curves for Relaxation Oscillations.

Let us choose $-\bar{\xi}=\xi=1$ with the other parameters as above so $-\bar{\zeta}=\zeta=-.5$. Then for $\alpha=1$ $(0\leq t\leq\pi)$

$$X = -.303\exp(.838t)\cos(t+.697) = \exp(x) - .5 \tag{101}$$

$$V = -.303\exp(.838t)[.838\cos(t+.697)-\sin(t+.697)] = v\exp(x) \tag{102}$$

while for $\bar{\alpha}=-1$ $(\pi\leq t\leq 2\pi)$

$$\bar{X} = -.303\exp[.838(t-\pi)]\cos(t+.697 = -[\exp(-\bar{x}) - .5] \tag{103}$$

$$\bar{V} = -.303\exp[.838(t-\pi)]\,[.838\cos(t+.697)-\sin(t+.697)] = \bar{v}\exp(-\bar{x}) \tag{104}$$

In Fig 11 the discontinuous X,V trajectory is shown by dashed curves while the x,v limit cycle is shown by a solid curve. Our choice of parameters made the square roots vanish in Eqs (85) and (86) and produced a semi-stable x,v limit cycle which attracts trajectories from outside itself but not from inside. A sample attracted trajectory is shown by a dashed curve in Fig 11. Also shown is a trajectory which begins at the origin and soon becomes unstable $(x\to\infty)$ as do other trajectories which start just inside the limit cycle and begin to approach the origin.

Suppose we begin as before but choose positive linear damping ($c=.838$) and negative quadratic damping ($\alpha=-1$ implying the first form of Eq (8) for our first arc and counterclockwise motion around the loop in Fig 1). Also let $F=0$ since this does not affect the x,v cycle. Then for $\alpha=-1$ $(0\leq t\leq\pi)$

$$X = -4.214\exp(-.838t)\cos(t-.697) = -[\exp(-x) - .5] \tag{105}$$

$$V = 4.214\exp(-.838t)\,[.838\cos(t-.697)+\sin(t-.697)] = v\exp(-x) \tag{106}$$

and for $\bar{\alpha}=1$ $(\pi\leq t\leq 2\pi)$

$$\bar{X} = -4.214\exp[-.838(t-\pi)]\cos(t-.697) = \exp(\bar{x}) - .5 \tag{107}$$

$$\bar{V} = 4.214\exp[-.838(t-\pi)]\,[.838\cos(t-.697)+\sin(t-.697)] = \bar{v}\exp(\bar{x}) \tag{108}$$

The trajectories in Fig 12 are similar to those in Fig 11 but are canted differently. Now the x,v limit cycle attracts from the inside but not from the outside as shown by sample dashed trajectories.

Of course in applications like that in Fig 2 there is no limit cycle.

ii. Complex x

Let us write $x=x_r+i\,x_i$, where $x_r$ and $x_i$ are the real and imaginary parts of x. In addition, let us keep all parameters real. Eq (4) then requires complex $X=X_r+iX_i$. For example, in the simple case with $\beta=F=0$, Eq (1) separates into real and imaginary parts with solutions

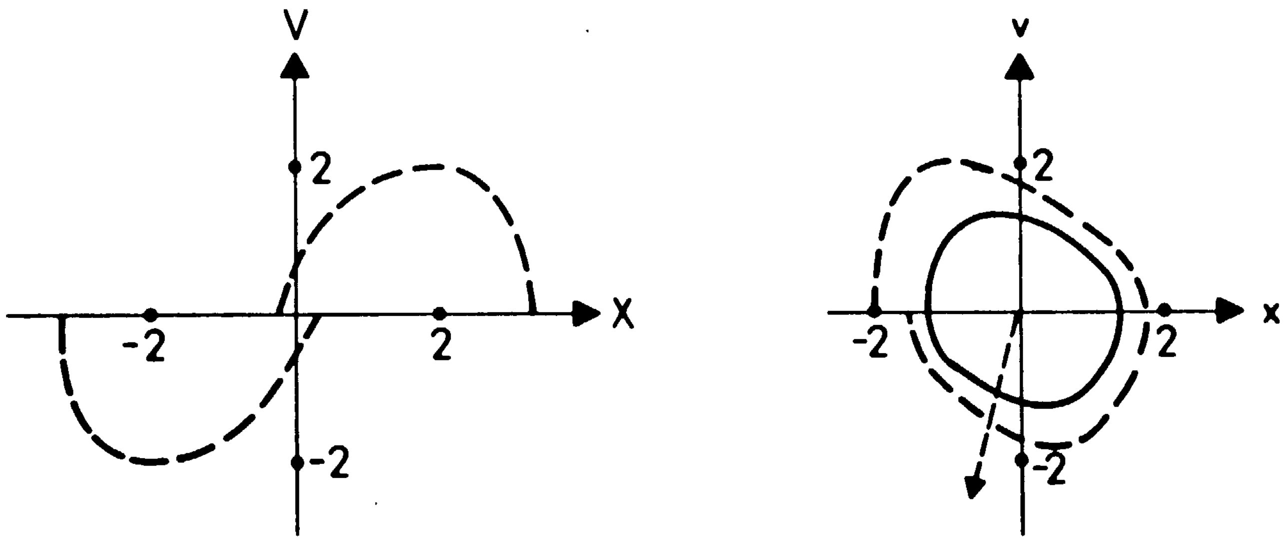


Figure 11. Half-stable Limit Cycle Attracting from Outside Only.

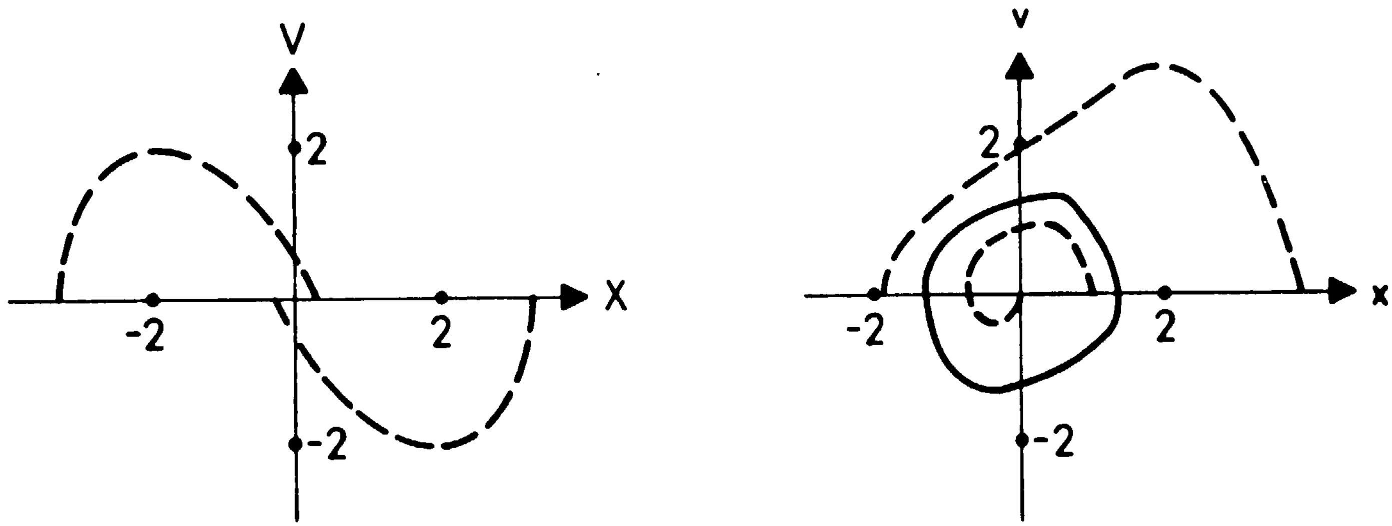


Figure 12. Half-stable Limit Cycle Attracting from Inside Only.

$$X_r = A_r\cos(\omega t-\theta_r) \tag{109}$$

$$X_i = A_i\cos(\omega t-\theta_i) \tag{110}$$

Even with linear damping and forcing Eq (1) separates into independent equations and both $X_r(t)$ and $X_i(t)$ are easily written down. However, even when $\beta=F=0$, Eq (5) separates into coupled equations

$$\ddot{x}_r + \alpha(\dot{x}_r^2 - \dot{x}_i^2) + (\omega^2/\alpha)[1+\delta\ \exp(-\alpha x_r)\cos\alpha x_i] = 0 \tag{111}$$

$$\ddot{x}_i + \alpha(2\dot{x}_r\dot{x}_i) + (\omega^2/\alpha)[1-\delta\ \exp(-\alpha x_r)\sin\alpha x_i] = 0 \tag{112}$$

The solutions to these equations are found by using $X_r$ and $X_i$ in Eq (4) and separating to find

$$\exp(\alpha x_r)\cos\alpha x_i = (A_r/\xi)\cos(\omega t-\theta_r) - \delta \tag{113}$$

$$\exp(\alpha x_r)\sin\alpha x_i = (A_i/\xi)\cos(\omega t-\theta_i) \tag{114}$$

By squaring and adding these equations we find $x_r(t)$ and by dividing one by the other we find $x_i(t)$. The same procedure may be followed even when $\beta$ and F are not zero, although the final expressions are more complicated. In general we now have four trajectory types: $(X_r,V_r)$, $(X_i,V_i)$, $(x_r,v_r)$ and $(x_i,v_i)$.

Consider the case in which the right sides of Eqs (87) and (88) are both real but negative, for example $\delta>0$ with $\delta^2\geq\delta/\bar{\delta}$. Since x must be complex these equations become

$$\exp(\alpha x_{r0L})[\cos\alpha x_{i0L}+i\ \sin\alpha x_{i0L}] = -\delta + \sqrt{\delta^2 - (\delta/\bar{\delta})} \tag{87}$$

$$\exp(\alpha x_{r1L})[\cos\alpha x_{i1L}+i\ \sin\alpha x_{i1L}] = -\delta - \sqrt{\delta^2 - (\delta/\bar{\delta})} \tag{88}$$

Since the right sides are real, we must have $\sin\alpha x_{iL}=0$ at both turning points. Since the right sides are negative, we must have $\cos\alpha x_{iL}=-1$ at both turning points. In other words, $\alpha x_{iL}$ must be an odd multiple of $\pi$ at turning points. Since Eq (114) will then vanish at turning points, we may choose $\theta_i=\pi/2$ if $A_i$ is not zero and if turning points occur whenever $\omega t$ is a

multiple of $\pi$, for example. If we begin at a turning point we can let $\theta_r=0$ in Eq (113) and compare with Eqs (87) and (88) to find $(A_r/\xi)=\sqrt{\delta^2 - (\delta/\bar{\delta})}$. In special cases like $\bar{\delta}=1/\delta$ or $\bar{\delta}=\delta=1$ this square root vanishes, and our choice of $\theta_r$ is arbitrary.

Let $\theta_r=0$ and $\theta_i=\pi/2$ in Eqs (113)and (114) with $\omega=\xi=\alpha=1$ for convenience. Squaring, adding and taking the derivative gives

$$\exp(2x_r) = A_i^2\sin^2 t + A_r^2\cos^2 t - 2A_r\delta\cos t + \delta^2 \quad (115)$$

$$v_r = \exp(-2x_r)[(A_i^2 - A_r^2)\cos t + A_r\delta]\sin t \quad (116)$$

while dividing and taking the derivative gives

$$\tan x_i = A_i\sin t/(A_r\cos t - \delta) \quad (117)$$

$$v_i = A_i\exp(-2x_r)\ (A_r - \delta\cos t) \quad (118)$$

when we use Eq (113) to simplify Eq (118). From the last two equations we see that for $A_i=0$, $v_i=0$ and $x_i$ is the infinite set of fixed points $\pm n\pi/\alpha$ with $n=0,1,2....$ For non-zero $A_i$, $\delta\geq A_r$ allows an infinite set of periodic solutions, while $\delta<A_r$ prevents $v_i$ from vanishing. From the first two equations we see that there can be only one periodic solution. Of course unstable trajectories of both types are allowed.

As a particular example consider $\bar{\delta}=\delta=\sqrt{2}$ with $-\bar{A}_r=A_r=\sqrt{\delta^2 - 1} = 1$ and $-\bar{A}_i= A_i=1$. Then for $\alpha=1$ ($0\leq t\leq\pi$)

$$\exp(2x_r) = 3 - 2\sqrt{2}\cos t \quad \text{and} \quad v_r = \sqrt{2}\exp(-2x_r)\ \sin t \quad (119)$$

$$\tan x_i = \sin t/(\cos t - \sqrt{2}) \quad \text{and} \quad v_i = \exp(-2x_r)(1-\sqrt{2}\cos t) \quad (120)$$

These equations describe the first segments of the nonlinear trajectories. Both linear trajectories are unit circles 90 degrees out of phase with each other in the case we are considering. For $\alpha=-1$, the second segments are

$$\exp(-2\bar{x}_r) = 3 + 2\sqrt{2}\cos t \quad \text{and} \quad \bar{v}_r = -\sqrt{2}\exp(2\bar{x}_r)\ \sin t \quad (121)$$

$$\tan\bar{x}_i = \sin t/(-\cos t - \sqrt{2}) \quad \text{and} \quad \bar{v}_i = \exp(2\bar{x}_r)\,(-1-\sqrt{2}\cos t) \tag{122}$$

The $(x_r,v_r)$ limit cycle is shown at the top of Fig 13 while three of the $(x_i,v_i)$ limit cycles are shown at the bottom, two as solid curves centered at $x_i = \pm\pi$ with clockwise motion of the representative point as for the real trajectory and one as a dashed curve centered at $x_i=0$ with motion in the opposite direction. The dashed trajectory and all those centered at even multiples of $\pi$ are forbidden since $\cos x_i=1$ instead of -1 as required when real turning points occur. The nonlinear real and imaginary trajectories are also out of phase. All of these limit cycles are bistable, attracting from inside as well as from outside. The imaginary trajectories in Fig 13 resemble the continents of Africa and South America.

Next, consider the case in which the right sides of Eqs (87) and (88) are both complex because $\delta^2<\delta/\bar{\delta}$. By equating real and imaginary parts we find

$$\exp(\alpha x_{r0L})\,\cos\alpha x_{i0L} = -\delta = \exp(\alpha x_{r1L})\,\cos\alpha x_{i1L} \tag{123}$$

$$\exp(\alpha x_{r0L})\,\sin\alpha x_{i0L} = \sqrt{\delta/\bar{\delta} - \delta^2} = -\exp(\alpha x_{r1L})\,\sin\alpha x_{i1L} \tag{124}$$

By squaring and adding these equations we find

$$\exp(2\alpha x_{r0L}) = \exp(2\alpha x_{r1L}) = \delta/\bar{\delta} \tag{125}$$

so $\delta$ and $\bar{\delta}$ must have the same sign and $x_{r0L}=x_{r1L}$. From Eq (124) we also see that $x_{i0L}=-x_{i1L}$. If we use Eq (125) in Eqs (123) and (124), we find

$$\cos\alpha x_{iL} = \pm\sqrt{\delta\bar{\delta}} \tag{126}$$

for $\delta<$ or $>0$. By comparing Eqs (113) and (123) we see that for $A_r\neq 0$ we can take $\bar{\vartheta}_r=\vartheta_r=\pi/2$ at turning points. By comparing Eqs (114) and (124) we see that we can take $\bar{\vartheta}_i=\vartheta_i=0$ with $(A_i/\xi)=\sqrt{\delta/\bar{\delta} - \delta^2}$.

Let $\vartheta_r=\pi/2$ and $\vartheta_i=0$ in Eqs (113) and (114) with $\omega=\xi=\alpha=1$ again. If we square , add and take the derivative we find

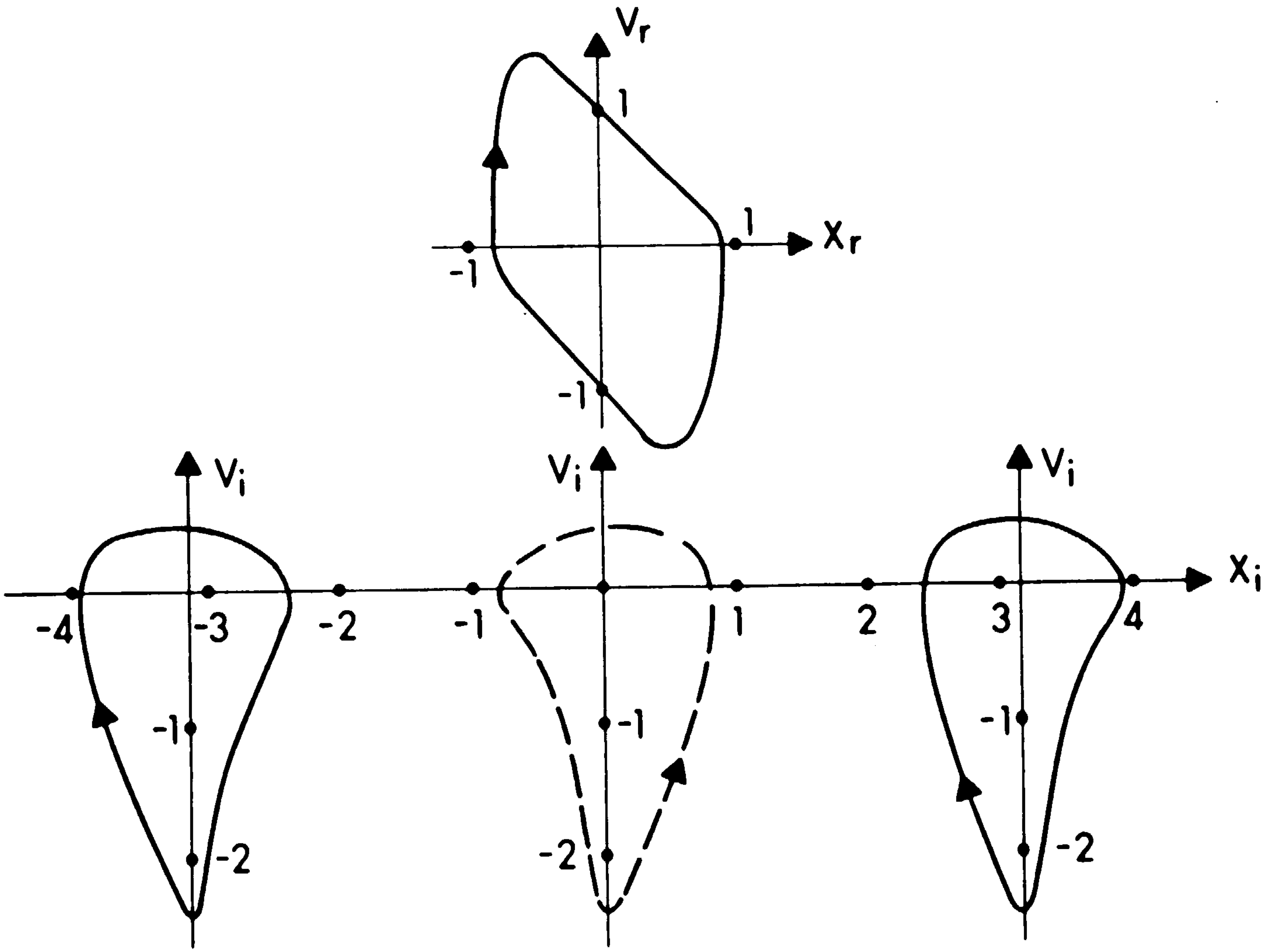


Figure 13. Complex Limit Cycles.

$$\exp(2x_r) = A_i^2\cos^2 t + A_r^2\sin^2 t - 2A_r\delta\sin t + \delta^2 \tag{127}$$

$$v_r = \exp(-2x_r)[(-A_i^2 + A_r^2)\sin t - A_r\delta]\cos t \tag{128}$$

while dividing and taking the derivative gives

$$\tan x_i = A_i\cos t/(A_r\sin t - \delta) \tag{129}$$

$$v_i = A_i\exp(-2x_r)(-A_r + \delta\sin t) \tag{130}$$

As a particular example consider $\bar{\delta}=\delta=1/\sqrt{2}$ with $\bar{A}_r=A_r=0$ and $-\bar{A}_i=A_i=\sqrt{1-.5} = 1/\sqrt{2}$. Then for $\alpha=1$ $(0\leq t\leq\pi)$

$$\exp(2x_r) = .5(1 + \cos^2 t) \quad \text{and} \quad v_r = -.5\exp(-2x_r)\sin t\cos t \tag{131}$$

$$\tan x_i = -\cos t \quad \text{and} \quad v_i = .5\exp(-2x_r)\sin t \tag{132}$$

For $\bar{\alpha}=-1$ $(\pi\leq t\leq 2\pi)$

$$\exp(-2x_r) = .5(1+\cos^2 t) \quad \text{and} \quad v_r = -.5\exp(2x_r)\sin t\cos t \tag{133}$$

$$\tan x_i = -\cos t \quad \text{and} \quad v_i = .5\exp(2x_r)\sin t \tag{134}$$

At the top of Fig 14 the $(x_r,v_r)$ trajectory is a double egg with pausing points ($v_r=0$ without a change in sign) at $x_{r0}=x_{r1}=0$. The turning points $x_r=\pm.347$ occur half way through each segment. Since Eq (125) is not satisfied, this is not a limit cycle. The imaginary trajectories at the bottom of the figure are not limit cycles either and the middle one does not meet the condition $\cos x_{iL}=-1/\sqrt{2}$ for $\delta>0$.

Of course there are many other cases possible besides the simple symmetric examples we have considered. As usual, our examples are meant to illustrate the method rather than exhaust the possibilities.

## IV. SINUSOIDAL PARAMETRIC EXCITATION

When $F=F_0\cos qt$ in Eq (1), the restoring force of Eq (6) includes sinusoidal parametric excitation

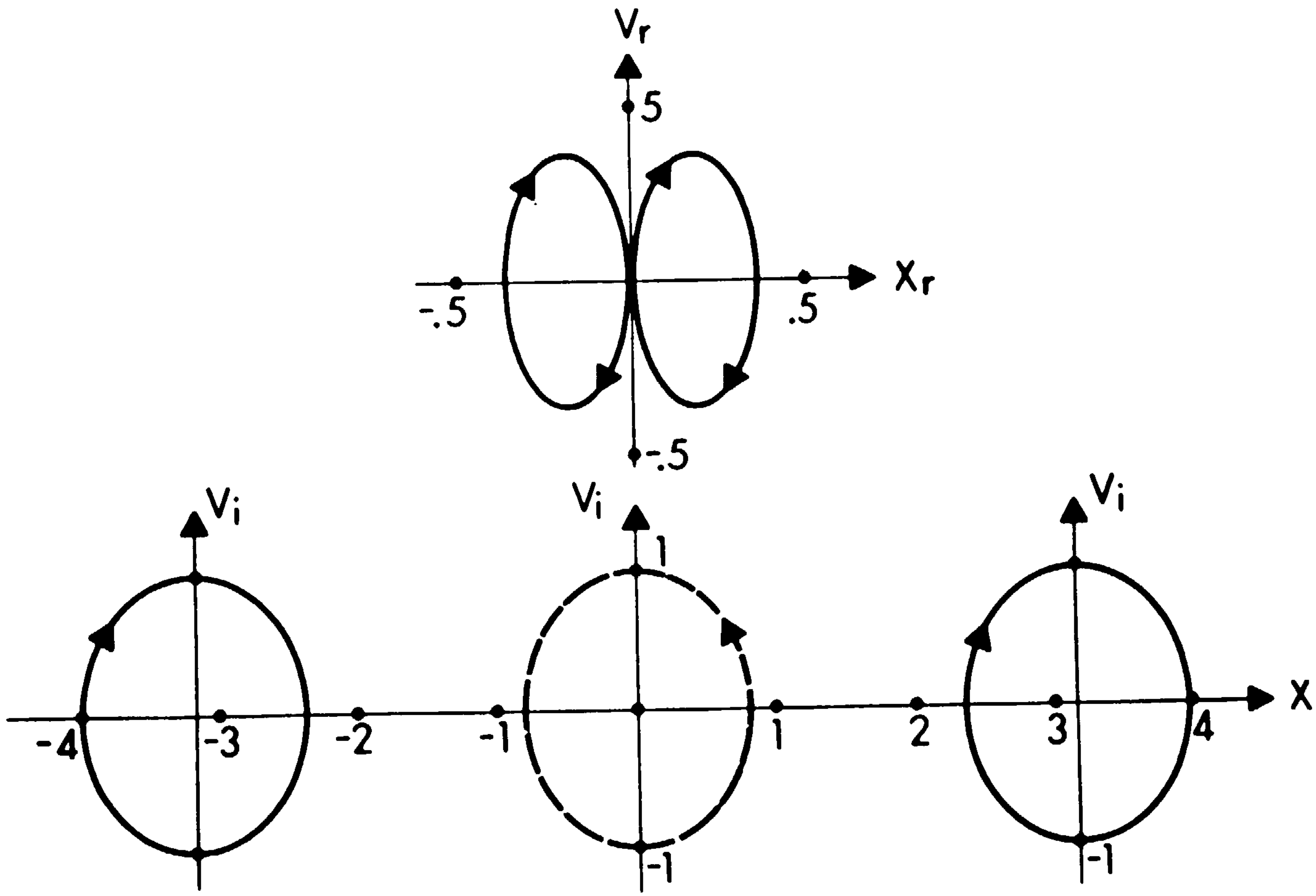


Figure 14. Complex Periodic Solutions without Attraction.

$$f = (\omega^2/\alpha)[1 + (1/\xi)[\zeta - \epsilon\cos qt]\exp(-\alpha x)\ ] \qquad (135)$$

As is well known, in the absence of linear damping X(t) is the superposition of free and forced oscillations and is periodic or aperiodic depending on whether $q/\omega$ is a rational or irrational number. Of course none of these solutions are limit cycles. When positive linear damping is present, we have the linear limit cycle of Eq (27) as the asymptotic solution upon which a nonlinear limit cycle may be built.Since

$$x = (1/\alpha)\ \ln[(X - \zeta)/\xi] \qquad (136)$$

from Eq (4), then for real x, $\zeta$ cannot be zero if X periodically assumes positive and negative values. The stability of x depends on the parameters $\zeta$ and $\xi$ as well as X. For $\xi>0$, x is stable provided $\zeta$ is less than the minimum value of X. For $\xi<0$, it is stable if $\zeta$ is greater than the maximum value of X. Once the parameter of X are given, we can determine ranges of $\zeta$ values which lead to stable or unstable x. For example, in Fig 15 we have plotted maximum and minimum values of X with zero damping and phase angle and $A=\omega=F_0=1$ so only q varies. The curves shown connect all irrational values of $q/\omega$ and enable us to obtain "safe" ranges of $\zeta$ which keep x stable. When $q=\omega$ X itself is unstable. For rational values of $q/\omega$ the maxima and/or minima of X may not coincide with the curves shown as indicated by asterisks when $q/\omega$ is 2 or 3. If $q/\omega$ is irrational and X is aperiodic the X,V trajectory will wander in a limited region of phase space[6]. In such a case the x,v trajectory might appear to be periodic for a time and then suddenly become unstable when $X=\zeta$. If $q/\omega$ is rational, X is periodic although the trajectory might be complicated and the period long. In such a case a stable x,v trajectory will also be periodic. However, if x(t) is observed for a time much less than the period, it might appear chaotic. In practice it is difficult to maintain frequency ratios constant for long times. In addition, damping is inevitable in real systems. Stability plots with some other parameter variable and the rest (including q) constant are easily constructed as well.

Eq (5) can be written as an autonomous system of three first order equations. For example, we may let $\dot{x}=v$ and $\dot{\psi}=1$ be two equations and

$$\dot{v} = -\alpha v^2 - \beta v - (\omega^2/\alpha)[\ 1 + (1/\xi)(\ \zeta + \epsilon\cos q\psi\ )\exp(-\alpha x)\ ] \qquad (137)$$

be our third equation. Some systems with three degrees of freedom, that is, three independent initial conditions $x_0$, $v_0$ and $\psi_0$, have been found to have chaotic as well as periodic solutions[10]. Since we have exact solutions for the parametrically excited exponential oscillator, we can distinguish periodic and aperiodic solutions but find no evidence for chaos. If Eq (5) is externally excited, its solutions will be based on solutions of a Hill's or Mathieu equation for X and chaos may appear. Since these solutions cannot be expressed by a finite number of elementary functions, they are beyond the limits we have set for ourselves here.

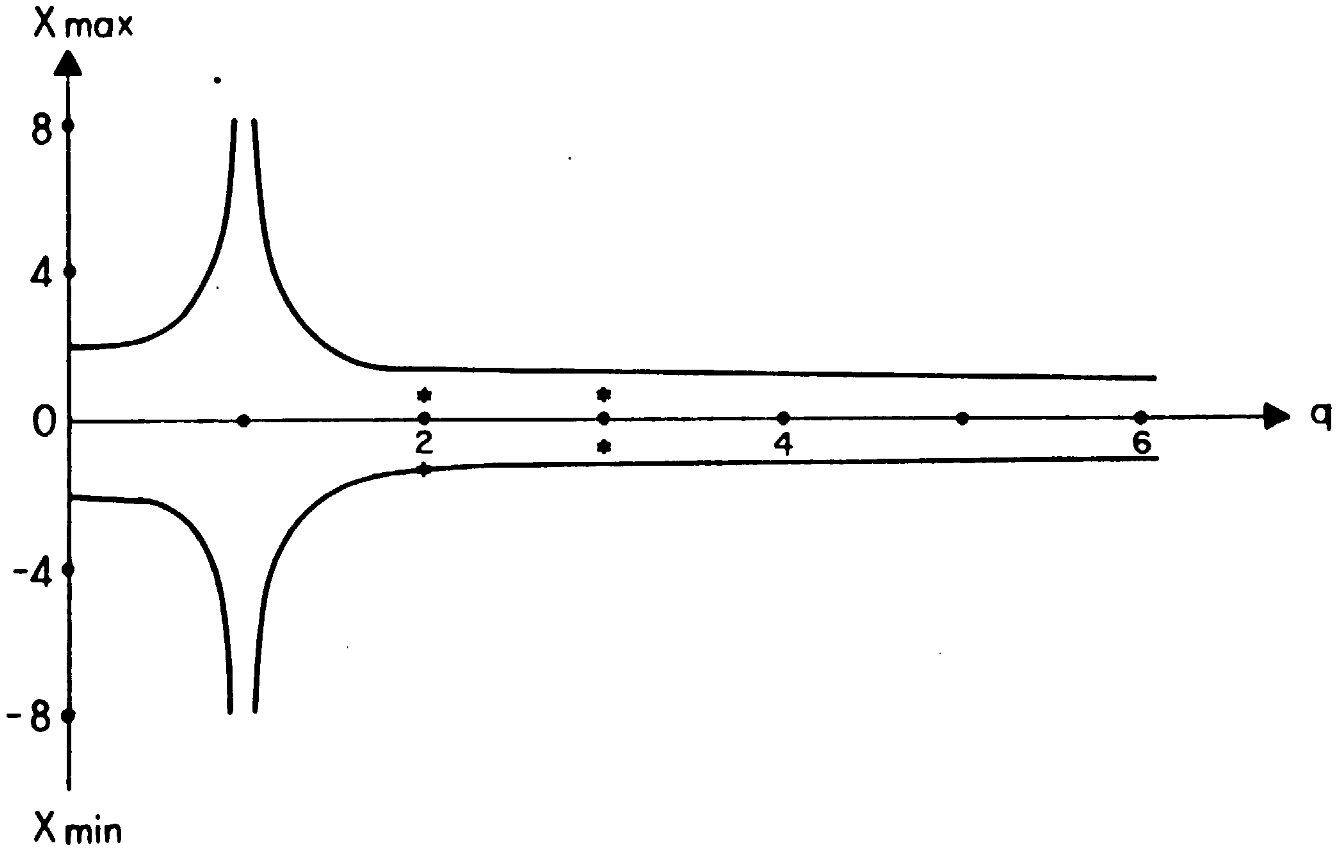


Figure 15. One Type of Stability Diagram for Parametric Sinusoidal Excitation.

So far in this section we have considered real forcing (external for X and parametric for x). If F is complex then X and x will also be complex and the method of finding x(t) based on Eq (20) is similar to that used in the previous section. Instead of Eq (135) the restoring force will now have a term containing parametric forcing by a traveling wave $\exp[i(qt - \alpha x_i)]$. If all the parameters are real and constant throughout the motion, then the asymptotic solutions are

$$X_r = L \cos(qt-\gamma) \tag{138}$$

$$X_i = L \sin(qt-\gamma) \tag{139}$$

where $L=F_0/\sqrt{(\omega^2-q^2)^2 + \beta^2 q^2}$ and $\tan\gamma=\beta q/(\omega^2-q^2)$. Thus we can find the asymptotic nonlinear solutions from

$$\xi \exp(\alpha x_r) \cos\alpha x_i = L\cos(qt-\gamma) - \zeta \tag{140}$$

$$\xi \exp(\alpha x_r) \sin\alpha x_i = L\sin(qt-\gamma) \tag{141}$$

From Eqs (138) and (139) we see that the limit cycle ellipses have the same amplitude and frequency but are ninety degrees out of phase. Since any parameters which allow stable oscillations will lead to limit cycles, there is no need to derive equations like Eqs (85) and (86). From Eqs (140) and (141) we find

$$[\xi \exp(\alpha x_r)]^2 = L^2 + \zeta^2 - 2L\zeta\cos\tau \tag{142}$$

$$\alpha v_r = qL\zeta \sin\tau/[\xi \exp(\alpha x_r)]^2 \tag{143}$$

$$\tan\alpha x_i = L \sin\tau / (L\cos\tau - \zeta) \tag{144}$$

$$\alpha v_i = qL(L - \zeta\cos\tau) / [\xi \exp(\alpha x_r)]^2 \tag{145}$$

where $\tau=(qt-\gamma)$. If $\zeta$ is zero, $x_r$ is a fixed point while $\alpha x_i=\tau=(qt-\gamma) \to \infty$ at constant speed. If $|\zeta|<|L|$, the $(x_r,v_r)$ trajectory is a limit cycle. However, $v_i$ never vanishes so the $x_i$ approach infinity with variable speed. If $|\zeta|=|L|$, $\alpha v_i=q/2$ independent of $\tau$ and all the $x_i$ approach infinity with constant speed. The $(x_r,v_r)$ trajectory is an oscillation between infinity and a point near the origin.. If $|\zeta|>|L|$, then there are an infinite number

of $(x_i, v_i)$ limit cycles centered at $x_i$ equal to multiples of $\pi/\alpha$ and one $(x_r, v_r)$ limit cycle which can be symmetric about the origin if $\zeta^2 = L^2 + \xi^2$.

For example, suppose $\alpha = L = \xi = q = 1$ and $\zeta = \sqrt{2}$. Eqs (142) and (143) become $\exp(2x_r) = 3 - 2\sqrt{2}\cos\tau$ and $v_r = \sqrt{2}\sin\tau/\exp(2x_r)$. Then for $\tau$ equal to even and odd multiples of $\pi$ we have $x_{r0} = -.881 = -x_{r1}$. Each imaginary limit cycle also has turning points located symmetrically on either side of their respective origins. However the maximum and minimum $v_i$ values are not equal in magnitude like the maximum and minimum $v_r$ values. The trajectories are easily sketched. All are allowed and have representative points which rotate clockwise.

## V. SUMMARY

In this report we have introduced the exponential oscillator and discussed some of its exact solutions. In particular we have noted that nonlinear limit cycles can exist whether or not linear limit cycles exist, at least in some cases. Examples were given restricted to real parameters but allowing complex variables. These were also limited to one or two segment motions. Included among these examples was an application involving mechanical hysteresis, namely a cyclic stress-strain curve. Clearly we have not exhausted the subject of the exponential oscillator. We merely mentioned the possiblilty of solutions involving more than two segments as well as solutions which cannot be expressed by a finite number of elementary functions. It is also clear that the oscillator we have discussed is merely one example of many nonlinear equations with exact solutions which can be generated by the same method. Not only can we use other forms besides Eq (4) in the transformation of Eq (3), we can also use other transformations of the dependent and/or independent variable singly or in succession as well as other starter equations. All of these matters are subjects for future work.

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